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Exact Solution of Space-Time Fractional Fokker-Planck Equations by Adomian Decomposition Method*

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Abstract: In the present paper, we solve a non-linear time-fractional and a linear space-time fractional Fokker–Planck equation (FPE) using Adomian decomposition method. The space and time fractional derivatives are considered in Caputo sense and the solutions are obtained in closed form, in terms of Mittag-Leffler functions.
Key words: Adomian decomposition method, Caputo fractional derivative, Mittag-Leffler function, Fokker-Planck equation, fractional differential equation.
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1. Introduction

The Fokker–Planck equation (FPE), first applied to investigate Brownian motion[1] and the diffusion mode of chemical reactions[2], is now largely employed, in various generalized forms, in physics, chemistry, engineering and biology[3]. The FPE arises in kinetic theory[4], where it describes the evolution of the one-particle distribution function of a dilute gas with long-range collisions, such as a Coulomb gas. For some applications of this equation one can refer the works of He and Wu[5], Jumarie[6], Kamitani and Matsuba[7], Xu et al.[8], and Zak[9]. The general FPE for the motion of a concentration field $u(x,t)$ of one space variable x at time t has the form[3]

(1.1) $$\frac{\partial u}{\partial t}=\left[-\frac{\partial}{\partial x}A(x)+\frac{\partial^2}{\partial x^2}B(x)\right]u(x,t),$$

with the initial condition given by

(1.2) $$u(x,0)=f(x), \qquad x\in\mathbb{R},$$

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where $B(x)>0$ is called the diffusion coefficient and $A(x)$ is the drift coefficient. The drift and diffusion coefficients may also depend on time. Mathematically, this equation is a linear second-order partial differential equation of parabolic type. Roughly speaking, it is a diffusion equation with an additional first-order derivative with respect to *x*.

There is a more general form of Fokker–Planck equation which is called the non-linear Fokker–Planck equation. The nonlinear Fokker–Planck equation has important applications in various areas such as plasma physics; surface physics, population dynamics, biophysics, engineering, neurosciences, nonlinear hydrodynamics, polymer physics, laser physics, pattern formation, psychology and marketing (see Frank[10] and references therein). In the case of one variable, the nonlinear FPE is written in the following form

(1.3) $$\frac{\partial u}{\partial t}=\left[-\frac{\partial}{\partial x}A(x,t,u)+\frac{\partial^2}{\partial x^2}B(x,t,u)\right]u(x,t),$$

with the initial condition given by

(1.4) $$u(x,0)=f(x), \qquad x\in\mathbb{R}.$$

Due to the vast range of applications of the FPE, a lot of work has been done to find the numerical solution of this equation. In this context, the works of Buet et al.[11], Harrison[12], Palleschi et al.[13] (1990), Vanaja[14], and Zorzano[15] are worth mentioning.

It has been observed that diffusion processes where the diffusion takes place in a highly nonhomogeneous medium, the traditional FPE may not be adequate[16,17]. The nonhomogeneities of the medium may alter the laws of Markov diffusion in a fundamental way. In particular, the corresponding probability density of the concentration field may have a heavier tail than the Gaussian density, and its correlation function may decay to zero at a much slower rate than the usual exponential rate of Markov diffusion, resulting in long-range dependence. This phenomenon is known as anomalous diffusion[18]. Fractional derivatives play key role in modeling particle transport in anomalous diffusion including the space fractional Fokker-Planck (advection-dispersion) equation describing Levy flights, the time fractional Fokker-Planck equation depicting traps, and the time-space fractional equation characterizing the competition between Levy flights and traps[19,20]. Different assumptions on this probability density function lead to a variety of space-time fractional Fokker-Planck equations.

The non-linear space-time fractional FPE can be written in the following general form

$$(1.5)\qquad \begin{aligned} &D_t^{\alpha}u = \left[-D_x^{\beta}A(x,t,u) + D_x^{2\beta}B(x,t,u)\right]u(x,t), \quad t>0, x>0,\\ &0<\alpha,\leq 1, 1<2\beta\leq 2 \end{aligned}$$

It can be obtained from the general Fokker-Planck equation by replacing the space and time derivatives by Caputo fractional derivatives D_t^{α} and D_x^{β} defined by (2.1). The function $u(x,t)$ is assumed to be a causal function of time and space, i.e., vanishing for $t<0$ and $x<0$. Particularly for $\alpha=1$ and $\beta=1$, the fractional FPE (1.5) reduces to the classical nonlinear FPE given by (1.3).

Recently several numerical methods have been proposed for solutions of space and/or time fractional Fokker-Planck equations[21-24]. In this paper we obtain closed form solutions of a linear space-time fractional and a non-linear time-fractional FPE using Adomian decomposition method. The Adomian decomposition method has been introduced and developed by Adomian[25,26]. It is useful for obtaining closed form or numerical approximation for a wide class of stochastic and deterministic problems in science and engineering. This method has further been modified by Wazwaz[27] and more recently by Luo[28] and Zhang and Luo[29]. A considerable amount of research work has been done recently in applying this method to a wide class of linear and nonlinear ordinary differential equations, partial differential equations, and integro-differential equations. For more details we refer Adomian[25, 26], Fokker[4], Frank[10], Mittal and Nigam[30], Wazwaz[31, 32], and the references there in.

2. Preliminaries

Caputo fractional derivative of order α for a function $f(x)$ with $x\in\mathbb{R}^{+}$ is defined as[33]

$$(2.1)\qquad \begin{aligned} D_x^{\alpha}f(x) &= \frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{f^{(m)}(\xi)}{(x-\xi)^{\alpha+1-m}}d\xi, (m-1<\alpha\leq m), m\in\mathbb{N},\\ &= J_x^{m-\alpha}D^m f(x). \end{aligned}$$

Here $D^m \equiv \dfrac{d^m}{dx^m}$ and J_x^{α} stands for the **Riemann-Liouville fractional integral operator** of order $\alpha>0$ defined as[34]

(2.2) $$J_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt, t>0, (m-1<\alpha\le m), m\in\mathbb{N}.$$

Clearly from the definition(2.1), we have

(2.3) $$D_x^{\alpha} x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} x^{\mu-\alpha}, \mu>-1.$$

For Riemann-Liouville fractional integral and Caputo fractional derivative, we have the following relation

(2.4) $$J_x^{\alpha} D_x^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}.$$

The **Mittag-Leffler function** which is a generalization of exponential function is defined as[35]:

(2.5) $$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} (\alpha\in\mathbb{C}, \mathrm{Re}(\alpha)>0),$$

Adomian decomposition method for nonlinear fractional partial differential equations[36]: We consider the nonlinear fractional partial differential equation written in an operator form as

(2.6) $$D_t^{\alpha} u(x,t) + Lu(x,t) + Nu(x,t) = g(x,t), x>0,$$

where D_t^{α} is Caputo fractional derivative of order $\alpha, m-1<\alpha\le m$, defined by equation (2.1), L is a linear operator which might include other fractional derivatives of order less than α, N is a non-linear operator which also might include other fractional derivatives of order less than α and $g(x,t)$ is source term.

We, apply the operator J_t^{α} to both sides of equation (2.6), use result (2.4) to obtain

(2.7) $$u(x,t) = \sum_{k=0}^{m-1} \left(\frac{\partial^k u}{\partial t^k}\right)_{t=0} \frac{t^k}{k!} + J_t^{\alpha} g(x,t) - J_t^{\alpha}\left[Lu(x,t) + Nu(x,t)\right],$$

Next, we decompose the unknown function u into sum of an infinite number of components given by the decomposition series

(2.8) $$u = \sum_{n=0}^{\infty} u_n,$$

and the nonlinear term is decomposed as follows

(2.9) $$Nu = \sum_{n=0}^{\infty} A_n,$$

where A_n are Adomian polynomials given by

(2.10) $$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}, n = 0,1,2,....$$

The components $u_0, u_1, u_2, ...$ are determined recursively by substituting (2.8) (2.9) into (2.7) leads to

(2.11) $$\sum_{n=0}^{\infty} u_n = \sum_{k=0}^{m-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{k!} + J_t^{\alpha} g(x,t) - J_t^{\alpha}\left[L\left(\sum_{n=0}^{\infty} u_n\right) + \sum_{n=0}^{\infty} A_n\right],$$

This can be written as

(2.12) $$u_0 + u_1 + u_2 + ... = \sum_{k=0}^{m-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{k!} + J_t^{\alpha} g(x,t) - J_t^{\alpha}\left[L(u_0 + u_1 + u_2 + ...) + (A_0 + A_1 + A_2 + ...)\right],$$

Adomian method uses the formal recursive relations as

(2.13) $$u_0 = \sum_{k=0}^{m-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{k!} + J_t^{\alpha} g(x,t),$$

$$u_{n+1} = -J_t^{\alpha}\left(L(u_n) + A_n\right), n \geq 0.$$

3. Solutions of Fractional Fokker-Planck Equations

Example.1 We consider the non-linear time fractional FPE[23]:

(3.1) $$D_t^{\alpha} u(x,t) = \left[-\frac{\partial}{\partial x}\left(\frac{4u}{x} - \frac{x}{3}\right) + \frac{\partial^2}{\partial x^2}(u)\right] u(x,t), \quad ,$$
$$t > 0,\ x > 0, 0 < \alpha \leq 1$$

where D_t^{α} is Caputo fractional derivative defined by (2.1) and initial condition is

(3.2) $$u(x,0) = x^2.$$

Applying J_t^{α} to both sides of equation (3.1), using result (2.4), we have

(3.3)
$$u=\sum_{k=0}^{1-1}\frac{t^k}{k!}\left[D_t^k u\right]_{t=0}+J_t^{\alpha}\left\{\frac{\partial}{\partial x}\left(\frac{xu}{3}\right)\right\}-J_t^{\alpha}\left[\left\{-\frac{\partial}{\partial x}\left(\frac{4}{x}\right)+\frac{\partial^2}{\partial x^2}\right\}u^2\right].$$

This gives the following recursive relations using equation (2.13),

(3.4)
$$u_0=\sum_{k=0}^{0}\frac{t^k}{k!}\left[D_t^k u\right]_{t=0},$$

(3.5)
$$u_{n+1}=J_t^{\alpha}\left\{\frac{\partial}{\partial x}\left(\frac{xu_n}{3}\right)\right\}-J_t^{\alpha}\left[A_n\right], n=0,1,2,...$$

where

(3.6)
$$A_n=\frac{1}{n!}\left[\left\{\frac{\partial}{\partial x}\left(\frac{4}{x}\right)-\frac{\partial^2}{\partial x^2}\right\}\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{n}\lambda^i u_i\right)^2\right]_{\lambda=0}, n=0,1,2,....$$

Which using results (2.3), (2.2) and (3.2) gives

(3.7)
$$u_0=x^2,$$

(3.8)
$$A_0=0,$$

(3.9)
$$u_1=\left(\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)x^2,$$

(3.10)
$$A_1=0,$$

(3.11)
$$u_2=\left(\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)x^2,$$

(3.12)
$$A_2=0,$$

(3.13)
$$u_3=\left(\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right)x^2,$$

and so on for other components.

Substituting $u_0, u_1, u_2, ...$ in equation (2.8) and making use of definition(2.5) , the solution of problem (3.1) is obtained as

(3.14)
$$u(x,t)=x^2 E_{\alpha}\left(t^{\alpha}\right).$$

Remark 1. *Setting* $\alpha=1$, *equation (3.1) reduces to non-linear FPE*[23]*:*

(3.15) $$\frac{\partial u}{\partial t}=\left[-\frac{\partial}{\partial x}\left(\frac{4u}{x}-\frac{x}{3}\right)+\frac{\partial^2}{\partial x^2}(u)\right]u(x,t), \quad , \quad t>0,\ x>0, 0<\alpha\le 1$$

with solution

(3.16) $$u(x,t)=x^2e^t.$$

Example 2. Consider the linear space-time fractional FPE:

(3.17) $$D_t^{\alpha}u(x,t)=\left[-D_x^{\beta}\left(px^{\beta}\right)+D_x^{2\beta}\left(qx^{2\beta}\right)\right]u(x,t), \quad t>0,\ x>0, \quad 0<\alpha\le 1, 1<2\beta\le 2,\ p,q\in\mathbb{R},$$

where $D_t^{\alpha}, D_x^{\beta}$ are Caputo fractional derivatives defined by (2.1) and initial condition is

(3.18) $$u(x,0)=x^{a-1}.$$

Applying J_t^{α} to both sides of equation (3.17), using result (2.4), we have

(3.19) $$u(x,t)=\sum_{k=0}^{1-1}\frac{t^k}{k!}\left[D_t^k u(x,t)\right]_{t=0}+J_t^{\alpha}\left[\left\{-D_x^{\beta}\left(px^{\beta}\right)+D_x^{2\beta}\left(qx^{2\beta}\right)\right\}u(x,t)\right].$$

This gives the following recursive relations using equation (2.13)

(3.20) $$u_0=\sum_{k=0}^{0}\frac{t^k}{k!}\left[D_t^k u(x,t)\right]_{t=0},$$

(3.21) $$u_{n+1}=J_t^{\alpha}\left[\left\{-D_x^{\beta}\left(px^{\beta}\right)+D_x^{2\beta}\left(qx^{2\beta}\right)\right\}u_n\right], n=0,1,2,\ldots$$

Which using results (2.3), (2.2) and (3.18), gives

(3.22) $$u_0=x^{a-1},$$

(3.23) $$u_1=\left(\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)bx^{a-1},$$

(3.24) $$u_2=\left(\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)b^2x^{a-1},$$

and so on for other components, where $b=q(a)_{2\beta}-p(a)_{\beta}$, where $(a)_{\beta}$ denotes the well known pochhammer symbol.

Substituting (3.22)-(3.24) in equation (2.8) and making use of definition (2.5), the solution of problem (3.17) is given by

(3.25) $$u(x,t)=x^{a-1}E_{\alpha}\left(bt^{\alpha}\right),b=q(a)_{2\beta}-p(a)_{\beta}.$$

Remark 1. *Setting* $\alpha=1$, *equation (3.17) reduces to linear space fractional FPE:*

(3.26) $$\frac{\partial u}{\partial t}=\left[-D_x^{\beta}\left(px^{\beta}\right)+D_x^{2\beta}\left(px^{2\beta}\right)\right]u(x,t),$$
$$t>0,\ x>0, 1<2\beta\le 2,$$

with solution

(3.27) $$u(x,t)=x^{a-1}e^{bt},b=q(a)_{2\beta}-p(a)_{\beta}.$$

Remark 2. *Setting* $\beta=1$, *equation (3.17) reduces to linear time fractional FPE:*

(3.28) $$D_t^{\alpha}u(x,t)=\left[-\frac{\partial}{\partial x}(px)+\frac{\partial^2}{\partial x^2}\left(qx^2\right)\right]u(x,t), \quad ,$$
$$t>0,\ x>0, 0<\alpha\le 1,\ p,q\in\mathbb{R}$$

with solution

(3.29) $$u(x,t)=x^{a-1}E_{\alpha}\left(bt^{\alpha}\right),b=qa^2+a(q-p).$$

Remark 3. *Setting* $\alpha=1,\beta=1$, *equation (3.17) reduces to linear FPE:*

(3.30) $$\frac{\partial u}{\partial t}=\left[-\frac{\partial}{\partial x}(px)+\frac{\partial^2}{\partial x^2}\left(qx^2\right)\right]u(x,t), \quad t>0,\ x>0,\ p,q\in\mathbb{R},$$

with solution

(3.31) $$u(x,t)=x^{a-1}e^{bt},b=qa^2+a(q-p).$$

Remark 4. *Setting* $\alpha=1,\beta=1,a=2,\ p=1,q=1/2$, *equation (3.17) reduces to linear FPE*[23]:

(3.32) $$\frac{\partial u}{\partial t}=\left[-\frac{\partial}{\partial x}(x)+\frac{\partial^2}{\partial x^2}\left(\frac{x^2}{2}\right)\right]u(x,t), \quad t>0,\ x>0,$$

with solution

(3.33) $$u(x,t)=xe^{t}.$$

Remark 5. *Setting* $\alpha = 1, \beta = 1, a = 3, p = 1/6, q = 1/12$, *equation (3.17) reduces to linear FPE*[23]*:*

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x}\left(\frac{x}{6}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{x^2}{12}\right) \right] u(x,t), \quad t > 0, x > 0, \tag{3.34}$$

with solution $u(x,t) = x^2 e^{t/2}$.

References

1. S. Chandrasekhar, Stochastic problems in physics and astronomy, *Reviews of Modern Physics*, **15 (**1943) 1–89.
2. H.A. Kramers, Brownian motion in a field of force and the diffusion model of chemical reactions, *Physica*, **7 (**1940) 284–304.
3. H. Risken, *The Fokker–Planck Equation: Method of Solution and Applications*, Berlin: Springer, 1989.
4. A. Fokker, The median energy of rotating electrical dipoles in radiation fields, *Annalen Der Physik*, **43** (1914) 810–820.
5. J.H. He, X.H. Wu, Construction of solitary solution and compacton-like solution by variational iteration method, *Chaos Solitons Fractals*, **29** (2006) 108–113.
6. G. Jumarie, Fractional Brownian motions via random walk in the complex plane and via fractional derivative, comparison and further results on their Fokker–Planck equations, *Chaos Solitons Fractals*, **22** (2004) 907–925.
7. Y. Kamitani, I. Matsuba, Self-similar characteristics of neural networks based on Fokker-Planck equation, *Chaos Solitons Fractals*, **20** (2004) 329–335.
8. Y. Xu, F.Y. Ren, J.R. Liang, W.Y. Qiu, Stretched Gaussian asymptotic behavior for fractional Fokker–Planck equation on fractal structure in external force fields, *Chaos Solitons Fractals***, 20** (2004) 581–586.
9. M. Zak, Expectation-based intelligent control, *Chaos Solitons Fractals*, **28** (2006) 616–626.
10. T.D. Frank, Stochastic feedback, nonlinear families of Markov processes, and nonlinear Fokker–Planck equations, *Physica A*, **331** (2004) 391–408.
11. C. Buet, S. Dellacherie, R. Sentis, Numerical solution of an ionic Fokker–Planck equation with electronic temperature, *SIAM J. Numer. Anal.*, **39** (2001) 1219–1253.
12. G. Harrison, Numerical solution of the Fokker–Planck equation using moving finite elements, *Numer. Methods Partial Differ. Eqns*, **4** (1988) 219–232.
13. V. Palleschi, F. Sarri, G. Marcozzi, M.R. Torquati, Numerical solution of the Fokker–Planck equation: a fast and accurate algorithm, *Phys. Lett. A*, **146** (1990) 378–386.
14. V. Vanaja, Numerical solution of simple Fokker–Planck equation, *Appl. Numer. Math.*, **9** (1992) 533–540.
15. M.P. Zorzano, H. Mais, L. Vazquez, Numerical solution of two-dimensional Fokker–Planck equations, *Appl. Math. Comput.*, **98** (1999) 109–117.
16. D.A. Benson, S.W. Wheatcraft, M.M. Meerschaert, Application of a fractional advection dispersion equation, *Water Resources Research*, 36**(6)** (2000) 1403–1412.

17. D.A. Benson, S.W. Wheatcraft, M.M. Meerschaert, The fractional-order governing equation of Levy motion, *Water Resources Research*, 36**(6)** (2000) 1413–1423.
18. J.P. Bouchaud, A. Georges, Anomalous diffusion in disordered media: statistical mechanisms, models and physical applications, *Physics Reports*, 195**(4-5)** (1990) 127–293.
19. R. Metzler, J. Klafter, Fractional Fokker-Planck equation: dispersive transport in an external force field, *Journal of Molecular Liquids*, 86**(1)** (2000) 219–228.
20. G. M. Zaslavsky, Chaos, fractional kinetics, and anomalous transport, *Physics Reports*, 371**(6)** (2002) 461–580.
21. Q. Yang, F. Liu, I. Turner, Computationally efficient numerical methods for time- and space-fractional Fokker-Planck equations, *Physica Scripta*, **T136** (2009) 014026(7pp).
22. Q. Yang, F. Liu, I. Turner, Stability and convergence of an effective numerical method for the time-space fractional Fokker-Planck equation with a nonlinear source term, *International Journal of Differential Equations* (2010), 464321(22pp).
23. A. Yildirim, Analytical approach to Fokker–Planck equation with space- and time-fractional derivatives by means of the homotopy perturbation method, *Journal of King Saud University (Science)*, **22** (2010) 257–264.
24. P. Zhuang, F. Liu, V. Anh, I. Turner, Numerical treatment for the fractional Fokker-Planck Equation, *Anziam J.*, **48** (2007) C759-C774.
25. G. Adomian, *Nonlinear Stochastic Operator Equations*, Academic Press, San Diego, 1986.
26. G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer, Boston, 1994.
27. A.M. Wazwaz, A reliable modification of Adomian decomposition method, *Appl. Math. Comput*, 92**(1)** (1998) 1-7.
28. X.G. Luo, A two-step Adomian decomposition method, *Appl. Math. Comput.*, 170**(1)** (2005) 570-583.
29. B.Q. Zhang, X.G. Luo, Q.B. Wu, The restrictions and improvement of the Adomian decomposition method, *Appl. Math. Comput.*, 171**(1)** (2006) 99-104.
30. R.C. Mittal, R. Nigam, Solution of fractional integro-differential equations by Adomian decomposition method, *Int. J. of Appl. Math. And Mech.*, 4**(2)** (2008) 87-94.
31. A.M. Wazwaz, *Partial Differential Equations: Methods and Applications*, Balkema Publishers, Leiden, 2002.
32. A.M. Wazwaz, *Partial Differential Equations and Solitary Waves Theory*, Higher Education Press, Beijing, 2009.
33. M. Caputo, *Elasticita e dissipazione*, Zanichelli, Bologma, 1969.
34. K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, 1993.
35. G.M. Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(x)$, *C. R. Acad. Sci. Paris*, **137** (1903) 554-558.
36. Z. Odibat, S. Momani, Numerical methods for nonlinear partial differential equations of fractional order, *Applied Mathematical Modelling*, **32** (2008) 28-39.