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Stability Analysis of a Prey-Predator Model with Beddington-DeAngelis Functional Response

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Abstract: In this paper, a spatial predator-prey system with Beddington - DeAngelis functional response and the modified Leslie - Gower type dynamics under homogeneous Neumann boundary condition is considered. The local and global asymptotic stability of the unique positive homogeneous steady state of the corresponding temporal model are discussed. Moreover, the local stability of the unique constant steady state of the spatiotemporal model is investigated and it is pointed out that spatial patterns cannot occur in the vicinity of this stable steady state.

Key words: Predator-Prey; Stability; Beddington-DeAngelis; Leslie-Gower; Reaction-Diffusion.

2010 Mathematics Subject Classification: 92D25

1. Introduction

The dynamical relationship between a predator and a prey has long been the dominant themes in mathematical ecology due to its universal existence and importance. One of the key components of the predator - prey system is the predator's functional response. The prey dependent functional response fail to model the interference among predators, and have been facing criticism from a section of biological and physiological communities. A more suitable general predator-prey theory should be based on the so called ratio-dependent theory, which asserts that the per capita predator growth rate should be a function of the ratio of prey to predator abundance.

On the other hand, in reality, prey and predator species are inhomogeneously distributed in different location of space at any given time. This consideration involve diffusion process which can be quite intricate as different concentration levels of prey and predator cause different population movements[1]. Thus, this movement or diffusion process

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must be incorporated in temporal ecological models that do not represent space explicitly.

The authors[2-5] have considered a diffusive prey-predator system with Leslie-Gower functional response with homogeneous Neumann boundary conditions and studied the local and global stability of the constant positive steady state, existence and non- existence of non-constant positive steady states.

The main aim of this paper is to study the stability behavior of the coexistence equilibrium point in the presence and absence of diffusion.

The organization of the paper is as follows. Section two devotes to the local and global asymptotical stability of the unique positive constant equilibrium point in the absence of diffusion. In section 3, the local stability of the unique constant positive equilibrium point in the presence of diffusion is presented.

2. Temporal Model

A prey - predator model with Beddington-DeAngelis functional response and modified Leslie-Gower dynamics in homogeneous environment is governed by the following system of non-linear ordinary differential equations

$$\text{(2.1)} \qquad \begin{aligned} \frac{dU}{d\tau} &= r_1\left(1-\frac{U}{K}\right)U - \frac{cU}{B+U+\varpi V}V, \\ \frac{dV}{d\tau} &= r_2\left(1-\frac{V}{s_1+sU}\right)V, \end{aligned}$$

subject to initial conditions $U(0)\geq 0$ and $V(0)\geq 0$. $U=U(\tau)$ and $V=V(\tau)$ represents the prey and predator densities respectively. The reaction parameters $r_1, K, c,\ B, \varpi, r_2, s, s_1$ are positive constants which stand for the intrinsic growth rate of the prey, the environmental carrying capacity of prey population, a maximum consumption rate, a saturation constant, predator interference, maximum per capita growth rate of the predator, the conversion factor of prey into predator, normalization constant respectively. Using the following scaling:

$$u=\frac{U}{K},\ v=\frac{V}{sK},\ \ t=r_1\tau,$$

and the parameters

$$\alpha = \frac{cs}{r_1}, \beta = \frac{B}{K}, \omega = s\varpi, b = \frac{s_1}{sK},$$

system (2.1) takes the non-dimensional form

$$(2.2)\qquad \begin{aligned} \frac{du}{dt} &= (1-u)u - \frac{\alpha u v}{\beta + u + \omega v} = G_1(u,v), \\ \frac{dv}{dt} &= \gamma\left(1 - \frac{v}{b+u}\right)v = G_2(u,v). \end{aligned}$$

It is easy to see that system (2.2) has four equilibrium points: $(0,0), (1,0), (0,b)$, and $E = (u, \tilde{v})$ where $\tilde{v} = b + u$ and u is the positive root of the quadratic equation

$$(1+\omega)(u)^2 + Bu + C = 0;\ B = \alpha + \beta + \omega b - (1+\omega),\ C = (\alpha - \omega)b - \beta,$$

which exists uniquely if

$$(2.3)\qquad (\alpha - \omega)b < \beta\ .$$

Theorem2.1: *The unique positive equilibrium point* $E = (u, \tilde{v})$ *is locally asymptotically stable if one of the following conditions is satisfied.*

$$(2.4)\qquad \begin{aligned} &i)\ b > \beta \text{ and } \frac{1}{b+\gamma} < \alpha < \frac{(\beta + \omega b)^2}{(b - \beta)}, \\ &ii)\ b \le \beta \text{ and } \alpha > \frac{1}{b+\gamma}. \end{aligned}$$

Proof: The Jacobian matrix at $E = (u, \tilde{v})$ is

$\mathbf{G_u}(E) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with

$$a_{11} = 1 - 2u - \frac{\alpha \tilde{v}(\beta + \omega \tilde{v})}{(\beta + u + \omega \tilde{v})^2},\quad a_{12} = -\frac{\alpha u(\beta + u)}{(\beta + u + \omega \tilde{v})^2},\ a_{21} = \gamma,\ a_{22} = -\gamma\ .$$

The trace of $\mathbf{G_u}(E)$

$$tr(\mathbf{G_u}(E)) = \frac{u\left(u^2 - (\alpha + 2)u + \left(1 - \alpha(b+\gamma)\right)\right) - \alpha b\gamma}{\alpha(b+u)}$$

is negative for $\alpha > \dfrac{1}{b+\gamma}$ and the determinant of $\mathbf{G_u}(E)$

$$\det\left(\mathbf{G_u}(E)\right) = \gamma\left(\frac{\alpha(-b+\beta)+(\beta+\omega b)^2+(1+\omega)\left(2(\beta+\omega b)+(1+\omega)u\right)u}{\left(\beta+u+\omega\tilde{v}\right)^2}\right)u,$$

is positive for $\beta \geq b \; or \; \beta < b$ and $\alpha < \dfrac{(\beta+\omega b)^2}{(b-\beta)}$.

Theorem 2.2: *If $\omega > \alpha$ then the local stability of system (2.2) ensures its global stability around the unique positive interior equilibrium point* $E = \left(u\,,\tilde{v}\right)$.

Proof: Let $\left(u(t), v(t)\right)$ be a positive solution of (2.2) and define a Dulac function

$$H = \frac{\beta+u+\omega v}{u^2\, v^2}.$$

From system (2.2), we have

$$\begin{aligned} Q &= \frac{\partial(H\,G_1)}{\partial u} + \frac{\partial(H\,G_2)}{\partial v} \\ &= -H(u,v)\frac{u^2+\beta+v(\omega-\alpha)}{u+\beta+\omega v} - H(u,v)\gamma\left(\frac{(u+\beta)(u+b)+\omega v^2}{(u+\beta+\omega v)(u+b)}\right)u. \end{aligned}$$

Therefore by Dulac criterion, we see that if $\omega > \alpha$ then system has no non-trivial positive periodic solutions. Thus the boundedness of the solutions of the system together with the assumption of local stability yields the global stability.

3. The spatiotemporal Model

In the predator-prey model (2.2), the prey and predator species are assumed to be spatially independent and dispersion of either population is not taken in to account. However, in reality, prey and predator populations are heterogeneously distributed over their habitat. Taking into account the mobility of the prey and predator population within a bounded habitat, the governing model (2.2) is modified as the following system of reaction-diffusion equations, after an appropriate scaling of spatial coordinates:

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + (1-u)u - \frac{\alpha u v}{\beta + u + \omega v}, \\ (3.1) \qquad &\frac{\partial v}{\partial t} = \delta \Delta v + \gamma\left(1 - \frac{v}{b+u}\right)v \end{aligned}$$

subject to the homogeneous Neumann boundary condition and non-negative initial condition. The operator Δ represents the Laplacian operator in two spatial domains and δ is the ratio of the diffusion coefficient of predator to prey.

It is straight forward to see that the positive unique equilibrium point $E = (u, \tilde{v})$ of the temporal model (2.2) is the constant positive unique equilibrium point of the spatiotemporal model (3.1). The local stability of this constant positive steady state $E = (u, \tilde{v})$ is discussed here.

Let $0 = \mu_0 < \mu_1 < \mu_2 < \mu_3 < \ldots$ be the eigen values of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition and φ_i be the i-th eigen function corresponding to the eigen value μ_i. The linearized system of (1.2) about E is

$$(3.2) \qquad \mathrm{u_t} = \mathrm{Lu},$$

where $\mathbf{D} = diag(1, \delta)$, $\mathbf{L} = \mathbf{D}\,\Delta + G_{\mathbf{u}}(E)$.

Now, expanding the solution $\mathbf{u}$ via $\mathbf{u} = \sum_{i=0}^{\infty} \psi_i(t)\varphi_i(x)$, where $\psi_i(t) \in \mathbb{R}_+^2$, substituting it in to (3.2) and equating the coefficients of each φ_i gives

$$\frac{d\varphi_i}{dt} = \mathbf{L}_i \varphi_i \ , \quad \mathbf{L}_i = -\mu_i\, \mathbf{D} + \mathbf{G}_{\mathbf{u}}(E)$$

Theorem 3.1: *If* $a_{11} < 0$ *then the constant positive steady state* $E = (u, \tilde{v})$ *of system (3.1) is uniformly asymptotically stable.*

Proof: The steady state E is stable if and only if $\mathbf{L}_i$ has two eigenvalues with negative real parts. The characteristics polynomial of $\mathbf{L}_i$ is given by

$$\varphi_i(\lambda) = \lambda^2 - tr(\mathbf{L}_i)\,\lambda + \det(\mathbf{L}_i),$$

where

$$tr\left(\mathbf{L}_i\right) = -\mu_i\left(\delta+1\right) + \mathrm{tr}\left(\mathbf{G}_{\mathbf{u}}\left(E\right)\right),$$
$$\det\left(\mathbf{L}_i\right) = \delta\mu_i^2 - \mu_i\left(a_{22}+\delta a_{11}\right) + \det\left(\mathbf{G}_{\mathbf{u}}\left(E\right)\right).$$

If $a_{11} < 0$ then $tr\left(\mathbf{L}_i\right) < 0$ and $\det\left(\mathbf{L}_i\right) > 0$. Thus, for each $i \geq 0$, both the roots $\lambda_{i,1}$, $\lambda_{i,2}$ of $\varphi_i\left(\lambda\right) = 0$ have negative real parts. Thus there exist some positive numbers κ_i such that $\mathrm{Re}\left\{\lambda_{i,1}\right\}, \mathrm{Re}\left\{\lambda_{i,2}\right\} \leq -\kappa_i \ \forall i.$

Let $\kappa = \min_i\left\{\kappa_i\right\}$, then, $\kappa > 0$ and $\mathrm{Re}\left\{\lambda_{i,1}\right\}, \mathrm{Re}\left\{\lambda_{i,2}\right\} \leq -\kappa \ \forall i.$

Hence, there exists a positive constant κ which is independent of i, such that $\mathrm{Re}\left\{\lambda_{i,1}\right\}, \mathrm{Re}\left\{\lambda_{i,2}\right\} \leq -\kappa \ \forall i.$ consequently, the spectrum of $\mathbf{L}$, which consists of eigen values, lies in $\left\{\mathrm{Re}\,\lambda \leq -\kappa\right\}$. Thus theorem 5.1.1 of Dan Henry[6] concludes the uniform asymptotical stability of E.

Remark 3.1. *As a consequence of theorem 3.1, if* a_{11} *is negative then diffusion cannot destabilize the stable constant coexistence steady state* E *of system (2.2) and hence Turing instability cannot occur in the vicinity of* E.

References

1. Yong-Hong Fana and Wan-Tong Li, Global asymptotic stability of a ratio-dependent predator–prey system with diffusion, *Journal of Computational and Applied Mathematics*, **188** (2006) 205–227.
2. Qunyi Bie and Rui Peng, Qualitative Analysis on a Reaction-Diffusion Prey-Predator Model and the Corresponding Steady-States**,** *Chin. Ann. Math,*. **30B** (2)(2009) 207-220.
3. Rui Peng and Ming Xin Wang, Qualitative analysis on a diffusive prey- predator model with ratio-dependent functional response, *Science in China Series A: Mathematics*, **51** (2008) 2043-2058.
4. Wonlyul Ko and Kimun Ryu, Non-constant positive steady-states of a diffusive predator-prey system in homogeneous environment, *J. Math. Anal. Appl.* , **327** (2007) 539-549.
5. H. B. Shi et al, Positive steady states of a diffusive predator-prey system with modified Holling-Tanner functional response, *Nonlinear Analysis: Real World Applications* , **11** (2010) 3711-3721.
6. D. Henry, *Geometric Theory of Semi linear Parabolic Equations, Lecture Notes in Mathematics*, vol. 840, Springer-Verlag, Berlin, 1981.