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On Generalized Measure of Weighted Symmetric Directed Divergence;Information Improvement

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Abstract: In the present paper we have obtained weighted directed divergence and symmetric directed divergence, information improvement, generalized information improvement corresponding to Sharma and Taneja[5] measure of entropy.
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1. Introduction

Let $P=(p_{1,}p_2,.......p_n)$ be a probability distribution and $W=(w_{1,}w_2,.......w_n)$ be a set of weights associated with the n- outcomes, then corresponding to Shannon's[1] measure of entropy,

(1.1) $$S(P)=-\sum_{i=1}^{n} p_i \ln p_i,$$

Guiasu[4] defined a measure of weighted entropy

(1.2) $$S(P,W)=-\sum_{i=1}^{w} w_i p_i \ln p_i,$$

and characterized it axiomatically. Now if $Q=(q_1,q_2,_{........}q_n)$ is another probability distribution then Kullback Leibler[2] gave a measure of directed divergence of P from Q, as

$$D(P:Q)=\sum_{i=1}^{n} p_i \ln\frac{p_i}{q_i}, \tag{1.3}$$

($\sum$ stands for summation over i from 1 to n through this paper) Kapur[7] has defined weighted directed divergence axiomatically. A measure $D(P:Q,W)$ will be said to be an appropriate measure of weighted directed divergence if

(i) It is a continuous function of $(p_{1,}p_2,....p_n)$; $(q_{1,}q_2,....q_n)$; $(w_{1,}w_2,....w_n)$.

(ii) It is permutationally symmetric, i.e. it does not change when the triplets $(p_{1,}q_1,w_1)$, $(p_{2,}q_2,w_2)$,.....(p_n,q_n,w_n), are permuted amongst themselves.

(iii) It is always ≥ 0 and vanishes when $p_i = q_i$, $i = 1,2,....n$.

(iv) It is a convex function of $(p_1, p_2,....p_n)$ which has its minimum value zero when $p_i = q_i$, $i = 1,2,....n$.

(v) It reduces to a positive multiple of an ordinary measure of directed divergence when all the weights are equal.

Taneja and Tuteja[6] gave the following weighted directed divergence corresponding to $D(P:Q)$ as

$$D_0(P:Q,W) = \sum_{i=1}^{n} w_i p_i \ln\frac{p_i}{q_i}, \tag{1.4}$$

But very soon Kapur[7] has pointed out that [1.4] is not a correct weighted directed divergence since it does not always satisfy (iii). Kapur[7] has suggested the following weighted directed divergence corresponding to Kullback Leibler's[2] measure

$$D_0(P:Q,W) = \sum_{i=1}^{n} w_i\left(p_i \ln\frac{p_i}{q_i} - p_i + q_i\right), \tag{1.5}$$

Kapur[7] also gave the following weighted divergence corresponding to Havrda-Charvat's[3] directed divergence as

(1.6)

$$D_{\alpha}(P:Q,W)=\frac{1}{\alpha(\alpha-1)}\sum_{i=1}^{n} w_i\left\{p_i^{\alpha}q_i^{1-\alpha}-\alpha p_i+\alpha q_i-q_i\right\}, \alpha>0.$$

It is obvious that

(1.7) $\qquad Lt_{\alpha\to 1} D_{\alpha}(P:Q,W)=D_0(P:Q,W),$

(1.8) $\qquad$ It is well known that $D(P:Q)\neq D(Q:P),$

(1.9) $\qquad$ So naturally $D(P:Q,W)\neq D(Q:P,W),$

Thus a weighted symmetric directed divergence has been defined as

(1.10) $\qquad J(P:Q,W)=D(P:Q,W)+D(Q:P,W),$

(1.11) $\qquad$ Therefore $J(P:Q,W)=J(Q:P,W),$

The measure $I(P:Q:R,W)$ of weighted information improvement in going from Q to R when the true distribution is P and the weighted function is $W(x)$ is defined by

(1.12) $\qquad I(P:Q:R,W)=D(P:Q,W)-D(P:R,W)$

This has the following properties:

(1.13) $\qquad$ when $Q=R$, $I(P:Q:R,W)=0,$
i.e. no improvement, intuitionally correct

(1.14) $\qquad$ when $Q=P$, $I(P:Q:R,W)<0,$
i.e., there cannot be any distribution better than true distribution.

(1.15) $\qquad$ when $R=P$, $I(P:Q:R,W)>0,$
i.e. definitely improvement if we reach to true distribution. Some times during the course of investigation true distribution P also changes to another distribution S. In this case generalized measure of weighted information improvement $G(P:S/Q/R,W)$ is defined as

(1.16) $\qquad G(P:S/Q/R,W)=\dfrac{I(P:Q:R,W)+I(S:Q:R,W)}{2},$

when there is no change in true distribution i.e. $S = P$.

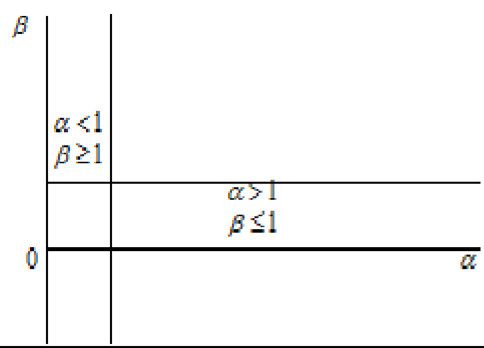
$G(P{:}S/Q/R,W) = I(P{:}Q{:}R,W)$, This is again intutionally correct.
In section 1.2 we will obtain a measure of weighted directed divergence with 2 parameters, which generalizes (1.6) and enumerate its certain properties. In section 1.3 we will obtain measure of weighted symmetric directed divergence, weighted information improvement and weighted generalized information improvement.

2. Generalized Weighted Directed Divergence

Now consider

(1.17)
$$D_{\alpha,\beta}(P:Q,W) = \frac{\sum w_i\left\{p_i^{\alpha} q_i^{1-\alpha} - p_i^{\beta} q_i^{1-\beta} - (\alpha-\beta)(p_i - q_i)\right\}}{(\alpha-\beta)(\alpha+\beta-1)},$$

where $\alpha, \beta \geq 0$ and $\alpha > 1, \beta \leq 1$ *or* $\alpha < 1, \beta \geq 1$
$\alpha = \beta = t$ not admissible except at t=1
Range of validity id given in the following figure.



Properties of $D_{\alpha,\beta}(P:Q,W)$

(1.18) (P_1) $D_{\alpha,\beta}(P{:}Q,W) = D_{\beta,\alpha}(P{:}Q,W)$

(P_2) $D_{\alpha,1}(P:Q,W) = D_{\alpha}(P:Q,W);$

(1.19) $D_{1,\beta}(P{:}Q,W) = D_{\beta}(P{:}Q,W)$

(P_3) $D_{\alpha,\beta}(P:Q,W) = 0$ *when* $P = Q,$

(P_4) $D_{\alpha,\beta}(P:Q,W)$

(1.20) $$= \frac{\alpha(\alpha-1)D_{\alpha}(P:Q,W)+\beta(1-\beta)D_{\beta}(P:Q,W)}{(\alpha-\beta)(\alpha+\beta-1)}$$

Since $\frac{\alpha(\alpha-1)}{(\alpha-\beta)(\alpha+\beta-1)}$ and $\frac{\beta(1-\beta)}{(\alpha-\beta)(\alpha+\beta-1)}$,
both are >0 for $\alpha<1, \beta\geq 1$ *or* $\alpha>1, \beta\leq 1$, so $D_{\alpha\beta}(P:Q,W)$ will be convex function of $p_1, p_2, ..., p_n$ (being sum of two convex functions of $p_1, p_2, ..., p_n$ with positive coefficients) (P_5) $D_{\alpha,\beta}(P:Q,W)$ has minimum value at $p_i=q_i \; \forall \; i=1,2,......,n,$ its minimum value is zero, so $D_{\alpha,\beta}(P:Q,W)\geq 0,$ (P_6) $D_{\alpha,\beta}(P:Q,W)$ Reduces to ordinary directed divergence corresponding to measure of entropy given by Sharma and Taneja[5].

3. Generalized Weighted Symmetric Directed Divergence, Weighted Information Improvement and Weighted Generalized Information Improvement

(1.21) $$J_{\alpha,\beta}(P:Q,W) = D_{\alpha,\beta}(P:Q,W) + D_{\alpha,\beta}(Q:P,W),$$

$$= \frac{\sum w_i \left\{ p_i^{\alpha} q_i^{1-\alpha} + q_i^{\alpha} p_i^{1-\alpha} - p_i^{\beta} q_i^{1-\beta} - q_i^{\beta} p_i^{1-\beta} \right\}}{(\alpha-\beta)(\alpha+\beta-1)}$$

(1.22) $$I_{a,b}(P:Q:R,W) = D_{a,b}(P:Q,W) - D_{a,b}(P:R,W),$$

$$= \frac{\sum w_i \left\{ p_i^{\alpha}\left(q_i^{1-\alpha} - r_i^{1-\alpha}\right) - p_i^{\beta}\left(q_i^{1-\beta} - r_i^{1-\beta}\right) - (\alpha-\beta)(r_i - q_i) \right\}}{(\alpha-\beta)(\alpha+\beta-1)}$$

$$G_{\alpha,\beta}(P:S/Q/R,W)=(1/2)[I_{\alpha,\beta}(P:Q:R,W)+I_{\alpha,\beta}(S:Q:R,W)]$$

$$\sum w_i \left\{ \begin{array}{l} \left(\frac{p_i^{\alpha}+s_i^{\alpha}}{2}\right)(q_i^{1-\alpha}-r_i^{1-\alpha}) \\ -\left(\frac{p_i^{\beta}+s_i^{\beta}}{2}\right)\frac{(q_i^{1-\beta}-r_i^{1-\beta})-(\alpha-\beta)(r_i-q_i)}{(\alpha-\beta)(\alpha+\beta-1)} \end{array} \right\},$$

$$=\sum_{i=1}^{n} w_i \frac{\left\{\left(\frac{p_i^{\alpha}+s_i^{\alpha}}{2}\right)(q_i^{1-\alpha}-r_i^{1-\alpha})-\left(\frac{p_i^{\beta}+s_i^{\beta}}{2}\right)\right\}(q_i^{1-\beta}-r_i^{1-\beta})-(\alpha-\beta)(r_i-q_i)}{(\alpha-\beta)(\alpha+\beta-1)}.$$

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