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The 4-dissection of Equivalent Continued Fractions

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Abstract: Some power series and infinite products can be expressed in the form of continued fractions. Equivalent continued fraction representations for the ratio of two hypergeometric series have also been established. Equivalent continued fraction representations for the ratio of infinite products have been deduced as special cases of these results.

The m-dissection of the power series $P=\sum_{n=0}^{\infty} a_n q^n$ is the represent- tation of P as $P=P_0+P_1+......+P_{m-1}$, where $P_k=\sum_{n=0}^{\infty} a_{mn+k} q^{mn+k}$

In this paper, an attempt has been made to obtain 4-dissection of equivalent continued fractions.

Keywords: Continued fraction, Hypergeometric series, Infinite products.

1. Introduction

In the beginning of 20th century the great Indian Mathematician Srinivasa Ramanujan discovered a series of results involving continued fractions.The oldest and the most famous theorem associated with Ramanujan's career is the Rogers Ramanujan continued fraction given by L J Rogers[1]

$$C(q)=1+\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}.................=\frac{\left(q^2,q^3;q^5\right)_\infty}{\left(q,q^4;q^5\right)_\infty}$$

The m-dissection of the power series $P=\sum_{n=0}^{\infty} a_n q^n$ is the representation of P as $P=P_0+P_1+..........+P_{m-1}$, where $P_k=\sum_{n=0}^{\infty} a_{mn+k} q^{mn+k}$ Andrews[2] and Hirschhorn[3] have given` the 2 dissection and 5 dissection of the continued fractions C(q) and C(q)$^{-1}$. Lewis et al[4] obtained a conjecture of Hirschhorn on4-dissection of Ramanujan's Continued Fraction.Dennis et al[5] gave equivalent continued fraction representations for ratio of infinite products.

$$S(q)=\frac{(q^3,q^5,q^8)_\infty}{(q,q^7;q^8)_\infty}$$

$$=1+\frac{q\ +q^2}{1+}\,\frac{q^4}{1+}\,\frac{q^3+q^5}{1+}\,\frac{q^8}{1+}\,\frac{q^5+q^{10}}{1+}\,\frac{q^{12}}{1+\ldots}$$

$$=1+\frac{q\ +q^2}{1-q+}\,\frac{q\ +q^4}{1-q+}\,\frac{q\ +q^6}{1-q+\ldots}$$

$$=1+q+\frac{q^2}{1+q^3+}\,\frac{q^4}{1+q^5+}\,\frac{q^5}{1+q^7+\ldots}$$

Here an attempt has been made to obtain the 4-dissection for above equivalent continued fractions $S(q)$ and its inverse $S(q)^{-1}$.

2. Notations

Suppose that $|q|<1$, where q is non-zero complex number, this condition ensures that all the infinite products that we use will converge. We will use the notation,

(2.1) $$(z;q)_\infty=\prod_{n=0}^{\infty}\left(1-zq^n\right),$$

(2.2) $$[z;q]_\infty=(z;q)_\infty\left(z^{-1}q;q\right)_\infty,$$ (for $z\neq 0$) and often we write

(2.3) $$[z_1,z_2,\ldots\ldots\ldots\ldots z_n;q]_\infty=[z_1;q]_\infty[z_2;q]_\infty\ldots\ldots\ldots\ldots[z_n;q]_\infty,$$

The following facts can be easily verified;

(2.4) $$\left[z^{-1};q\right]_\infty=-z^{-1}[z;q]_\infty=[zq;q]_\infty,$$

(2.5) $$\left[z,zq;q^2\right]_\infty=[z;q]_\infty,$$

(2.6) $$[z,-z;q]_\infty=\left[z^2;q^2\right]_\infty,$$

(2.7) $$\left[z^{-1}q;q\right]_\infty=\left[z\ ;q\ \right]_\infty,$$

(2.8) $$\left[-1;q\right]_\infty \left[q;q^2\right]_\infty = 2.$$

Also, we have the following general relations;
Suppose $a_1, a_2 \ldots\ldots\ldots\ldots\ldots, a_n; b_1, b_{2,} \ldots\ldots\ldots\ldots\ldots, b_n \in C\setminus\{0\}$ satisfy

i) $a_i \neq q^n a_j$ for $i \neq j$ and any $n \in z$,

ii) $a_1 a_2 \ldots\ldots\ldots\ldots a_n = b_1 b_2 \ldots\ldots\ldots\ldots\ldots\ldots b_n$,

Then

(2.9) $$\sum_{i=1}^{n} \frac{\prod_{j=1}^{n}\left[a_i^{-1}b_j;q\right]_\infty}{\prod_{j=1,\, j\neq i}^{n}\left[a_i^{-1}a_j;q\right]_\infty} = 0,$$

This theorem appears without proof as given by Slater[6] and with a proof as given by Lewis[7].

3. Main Results

(3.1) $$\begin{aligned} S(q) = {} & \frac{\left[q^2,q^{10};q^{16}\right]_\infty}{\left[q^8;q^{16}\right]_\infty} \frac{\left[q^6;q^{32}\right]_\infty}{\left[q^2,q^4,q^{16};q^{32}\right]_\infty} \\ & + \frac{q^2\left[q^2,q^6;q^{16}\right]_\infty}{\left[q^8;q^{16}\right]_\infty} \frac{\left[q^{22};q^{32}\right]_\infty}{\left[q^4,q^{16},q^{18};q^{32}\right]_\infty} \\ & + \frac{q^2\left[q\ ,q^4;q^{16}\right]_\infty \left[q^{26};q^{32}\right]_\infty}{\left[q^7;q^{16}\right]_\infty \left[q^2,q^8,q^{16};q^{32}\right]_\infty} \\ & + \frac{\left[q^4,q^{17};q^{16}\right]_\infty \left[q^4,q^{10};q^{32}\right]_\infty}{q^2\left[q^7;q^{16}\right]_\infty \left[q^2,q^8,q^{16},q^{36};q^{32}\right]_\infty}, \end{aligned}$$

(3.2) $$\begin{aligned} S(q)^{-1} = {} & \frac{\left[q^2,q^{10};q^{16}\right]_\infty \left[q^{18};q^{32}\right]_\infty}{\left[q^8;q^{16}\right]_\infty \left[q^4,q^{16},q^{22};q^{32}\right]_\infty} + \frac{q^2\left[q^2,q^6;q^{16}\right]_\infty \left[q^2;q^{32}\right]_\infty}{\left[q^8;q^{16}\right]_\infty \left[q^4,q^6,q^{16};q^{32}\right]_\infty} \\ & - \frac{q^2\left[q,q^4;q^{16}\right]_\infty \left[q^{18};q^{32}\right]_\infty}{\left[q^7;q^{16}\right]_\infty \left[q^8,q^{16},q^{22};q^{32}\right]_\infty} - \frac{\left[q^4,q^{17};q^{16}\right]_\infty \left[q^4,q^{18};q^{32}\right]_\infty}{q^2\left[q^7;q^{16}\right]_\infty \left[q^6,q^8,q^{16},q^{36};q^{32}\right]_\infty} \end{aligned}$$

4. Proof

To prove (3.1) and (3.2) by making use of (2.1),(2.2) and (2.5)

(4.1) $S(q)=\dfrac{\left(q^3,q^5;q^8\right)_\infty}{\left(q,q^7;q^8\right)_\infty}$, can be written as $S(q)=\dfrac{\left[q^3,q^{11};q^{16}\right]_\infty}{\left[q,q^9;q^{16}\right]_\infty}$,

Now, setting $\left(a_1,a_2,a_3,a_4;b_1,b_2,b_3,b_4\right)=\left(1,-1,q,q^9;q^3,q^{11},-q^{-2},q^{-2}\right)$ and taking q^{16} for q in (2.9), we get

$$\frac{\left[q^3,q^{11},-q^{-2},q^{-2};q^{16}\right]_\infty}{\left[-1,q,q^9;q^{16}\right]_\infty}+\frac{\left[-q^3,-q^{11},q^{-2},-q^{-2};q^{16}\right]_\infty}{\left[-1,-q,-q^9;q^{16}\right]_\infty}+$$
$$\frac{\left[q^2,q^{10},-q^{-3},q^{-3};q^{16}\right]_\infty}{\left[q^{-1},-q^{-1},q^8;q^{16}\right]_\infty}+\frac{\left[q^{-6},q^2,-q^{-11},q^{-11};q^{16}\right]_\infty}{\left[q^{-9},-q^{-9},q^{-8};q^{16}\right]_\infty}=0,$$

By applying (2.4) to (2.8) we get

$$\frac{\left[q^3,q^{11};q^{16}\right]_\infty}{\left[q,q^9;q^{16}\right]_\infty}+\frac{\left[-q^3,-q^{11};q^{16}\right]_\infty}{\left[-q,-q^9;q^{16}\right]_\infty}=\frac{2}{\left[q^4,q^{16};q^{32}\right]_\infty\left(-q^{-4}\right)}$$
$$\left[-\frac{\left[q^2,q^{10};q^{16}\right]_\infty}{\left[q^8;q^{16}\right]_\infty}\frac{\left[q^6;q^{32}\right]_\infty}{\left[q^2;q^{32}\right]_\infty}\left(q^{-4}\right)-\frac{\left[q^2,q^6;q^{16}\right]_\infty}{\left[q^8;q^{16}\right]_\infty}\frac{\left[q^{22};q^{32}\right]_\infty}{\left[q^{18};q^{32}\right]_\infty}q^{-2}\right],$$

$$S(q)+S(-q)=\frac{2\left[q^2,q^{10};q^{16}\right]_\infty\left[q^6;q^{32}\right]_\infty}{\left[q^8;q^{16}\right]_\infty\left[q^2,q^4,q^{16};q^{32}\right]_\infty}$$
(4.2)
$$+\frac{2q^2\left[q^2,q^6;q^{16}\right]_\infty\left[q^{22};q^{32}\right]_\infty}{\left[q^8;q^{16}\right]_\infty\left[q^4,q^{16},q^{18};q^{32}\right]_\infty},$$

$$\alpha_1\left(q^2\right)=\frac{1}{2}\left[S(q)+S(-q)\right]=\frac{\left[q^2,q^{10};q^{16}\right]_\infty\left[q^6;q^{32}\right]_\infty}{\left[q^8;q^{16}\right]_\infty\left[q^2,q^4,q^{16};q^{32}\right]_\infty}$$
(4.3)
$$+\frac{q^2\left[q^2,q^6;q^{16}\right]_\infty\left[q^{22};q^{32}\right]_\infty}{\left[q^8;q^{16}\right]_\infty\left[q^4,q^{16},q^{18};q^{32}\right]_\infty},$$

again setting

$\left(a_1,a_2,a_3,a_4;b_1,b_2,b_3,b_4\right)=\left(1,-1,q^{14},-q^{18};q^{11},q^{13},-q,-q^7,\right)$ and taking q^{16} for q in (2.9), we get

$$\frac{\left[q^{11},q^{13},-q,-q^{7};q^{16}\right]_\infty}{\left[-1,q^{14},-q^{18};q^{16}\right]_\infty}+\frac{\left[-q^{11},-q^{13},q\ ,q^{7};q^{16}\right]_\infty}{\left[-1,-q^{14},q^{18};q^{16}\right]_\infty}$$
$$+\frac{\left[q^{-3},q^{-1},-q^{-13},-q^{-7};q^{16}\right]_\infty}{\left[q^{-14},-q^{-14},-q^{4};q^{16}\right]_\infty}$$
$$+\frac{\left[-q^{-7},-q^{-5},q^{-17},q^{-11};q^{16}\right]_\infty}{\left[-q^{-18},q^{-18},-q^{-4};q^{16}\right]_\infty}=0.$$

By applying (2.4) and (2.7) we get

$$\frac{q^2}{\left[-1,q^2,-q^2;q^{16}\right]_\infty}\left[\left[q^{11},q^3,-q,-q^9;q^{16}\right]_\infty-\left[-q^{11},-q^3,q,q^9;q^{16}\right]_\infty\right]$$
$$-\frac{q^4\left[q^3,q,-q^{13},-q^7;q^{16}\right]_\infty}{\left[q^{14},-q^{14},-q^4;q^{16}\right]_\infty}-\frac{\left[-q^7,-q^5,q^{17},q^{11};q^{16}\right]_\infty}{\left[-q^{18},q^{18},-q^4;q^{16}\right]_\infty}=0$$

$$\left[\left[q^3,q^{11},-q\ ,-q^9;q^{16}\right]_\infty-\left[-q^{11},-q^3,q\ ,q^9;q^{16}\right]_\infty\right]$$
$$=\left[\begin{array}{l}\dfrac{q^4\left[q\ ,q^3,q^4;q^{16}\right]_\infty\left[q^{14},q^{26};q^{32}\right]_\infty}{\left[q^7,q^{13};q^{16}\right]_\infty\left[q^8,q^{28};q^{32}\right]_\infty}\\ +\dfrac{\left[q^4,q^{11},q^{17};q^{16}\right]_\infty\left[q^{10},q^{14};q^{32}\right]_\infty}{\left[q^5,q^7;q^{16}\right]_\infty\left[q^8,q^{36};q^{32}\right]_\infty}\end{array}\right]\frac{2\left[q^4;q^{32}\right]_\infty}{q^2\left[q^{16};q^{32}\right]_\infty}.$$

Dividing by $\left[-q,q,q^9,-q^9;q^{16}\right]_\infty$ and using (2.3) to (2.7) we get,

$$\left[\frac{\left[q^3,q^{11};q^{16}\right]_\infty}{\left[q,q^9;q^{16}\right]_\infty}-\frac{\left[-q^3,-q^{11};q^{16}\right]_\infty}{\left[-q,-q^9;q^{16}\right]_\infty}\right]$$
$$=\frac{2q^2\left[q\ ,q^3,q^4;q^{16}\right]_\infty\left[q^4,q^{14},q^{26};q^{32}\right]_\infty}{\left[q^7,q^{13};q^{16}\right]_\infty\left[q^2,q^8,q^{16},q^{18},q^{28};q^{32}\right]_\infty}$$
$$+\frac{2\left[q^4,q^{11},q^{17};q^{16}\right]_\infty\left[q^4,q^{10},q^{14};q^{32}\right]_\infty}{q^2\left[q^5,q^7;q^{16}\right]_\infty\left[q^2,q^8,q^{16},q^{18},q^{36};q^{32}\right]_\infty},$$

By applying (2.2) we get

$$S(q)-S(-q)$$

$$(4.4)\qquad =\frac{2q^2\left[q,q^4;q^{16}\right]_\infty\left[q^{26};q^{32}\right]_\infty}{\left[q^7;q^{16}\right]_\infty\left[q^2,q^8,q^{16};q^{32}\right]_\infty}+\frac{2\left[q^4,q^{17};q^{16}\right]_\infty\left[q^4,q^{10};q^{32}\right]_\infty}{q^2\left[q^7;q^{16}\right]_\infty\left[q^2,q^8,q^{16},q^{36};q^{32}\right]_\infty}$$

$$(4.5)\qquad \beta_1(q^2)=\frac{1}{2}\left[S(q)-S(-q)\right]$$

$$=\frac{q^2\left[q,q^4;q^{16}\right]_\infty\left[q^{26};q^{32}\right]_\infty}{\left[q^7;q^{16}\right]_\infty\left[q^2,q^8,q^{16};q^{32}\right]_\infty}+\frac{\left[q^4,q^{17};q^{16}\right]_\infty\left[q^4,q^{10};q^{32}\right]_\infty}{q^2\left[q^7;q^{16}\right]_\infty\left[q^2,q^8,q^{16},q^{36};q^{32}\right]_\infty}$$

By using (4.3) and (4.5), we get (3.1)

$$S(q)=\frac{\left[q^2,q^{10};q^{16}\right]_\infty}{\left[q^8;q^{16}\right]_\infty}\frac{\left[q^6;q^{32}\right]_\infty}{\left[q^2,q^4,q^{16};q^{32}\right]_\infty}+\frac{q^2\left[q^2,q^6;q^{16}\right]_\infty}{\left[q^8;q^{16}\right]_\infty}\frac{\left[q^{22};q^{32}\right]_\infty}{\left[q^4,q^{16},q^{18};q^{32}\right]_\infty}$$

$$+\frac{q^2\left[q\ ,q^4;q^{16}\right]_\infty\left[q^{26};q^{32}\right]_\infty}{\left[q^7;q^{16}\right]_\infty\left[q^2,q^8,q^{16};q^{32}\right]_\infty}+\frac{\left[q^4,q^{17};q^{16}\right]_\infty\left[q^4,q^{10};q^{32}\right]_\infty}{q^2\left[q^7;q^{16}\right]_\infty\left[q^2,q^8,q^{16},q^{36};q^{32}\right]_\infty}.$$

Now, to prove (3.2), we consider (4.2)

$$S(q)+S(-q)=\frac{2\ \left[q^2,q^{10};q^{16}\right]_\infty\left[q^6;q^{32}\right]_\infty}{\left[q^8;q^{16}\right]_\infty\left[q^2,q^4,q^{16};q^{32}\right]_\infty}+\frac{2q^2\left[q^2,q^6;q^{16}\right]_\infty\left[q^{22};q^{32}\right]_\infty}{\left[q^8;q^{16}\right]_\infty\left[q^4,q^{16},q^{18};q^{32}\right]_\infty}.$$

Multiplying (4.2) by $\frac{\left[q^2,q^{18};q^{32}\right]_\infty}{\left[q^6,q^{22};q^{32}\right]_\infty}$ we get,

$$(4.6)\qquad \frac{\left[-q\ ,-q^9;q^{16}\right]_\infty}{\left[-q^3,-q^{11};q^{16}\right]_\infty}+\frac{\left[q\ ,q^9;q^{16}\right]_\infty}{\left[q^3,q^{11};q^{16}\right]_\infty}$$

$$=\frac{2\left[q^2,q^{10};q^{16}\right]_\infty\left[q^2,q^6,q^{18};q^{32}\right]_\infty}{\left[q^8;q^{16}\right]_\infty\left[q^2,q^4,q^6,q^{16},q^{22};q^{32}\right]_\infty}+\frac{2q^2\left[q^2,q^6;q^{16}\right]_\infty\left[q^2,q^{18},q^{22};q^{32}\right]_\infty}{\left[q^8;q^{16}\right]_\infty\left[q^4,q^6,q^{16},q^{18},q^{22};q^{32}\right]_\infty},$$

$$\alpha_2\left(q^2\right)=\frac{1}{2}\left[S(q)^{-1}+S(-q)^{-1}\right]$$

(4.7)
$$=\frac{\left[q^2,q^{10};q^{16}\right]_\infty\left[q^{18};q^{32}\right]_\infty}{\left[q^8;q^{16}\right]_\infty\left[q^4,q^{16},q^{22};q^{32}\right]_\infty}+\frac{q^2\left[q^2,q^6;q^{16}\right]_\infty\left[q^2;q^{32}\right]_\infty}{\left[q^8;q^{16}\right]_\infty\left[q^4,q^6,q^{16};q^{32}\right]_\infty}.$$

Now we consider (4.4)

$$S(q)-S(-q)$$
$$=\frac{2q^2\left[q,q^4;q^{16}\right]_\infty\left[q^{26};q^{32}\right]_\infty}{\left[q^7;q^{16}\right]_\infty\left[q^2,q^8,q^{16};q^{32}\right]_\infty}+\frac{2\left[q^4,q^{17};q^{16}\right]_\infty\left[q^4,q^{10};q^{32}\right]_\infty}{q^2\left[q^7;q^{16}\right]_\infty\left[q^2,q^8,q^{16},q^{36};q^{32}\right]_\infty}$$

Multiplying (4.4) by $\frac{\left[q^2,q^{18};q^{32}\right]_\infty}{\left[q^6,q^{22};q^{32}\right]_\infty}$ we get,

$$S(q)^{-1}-S(-q)^{-1}$$
$$=-\frac{2q^2\left[q,q^4;q^{16}\right]_\infty\left[q^2,q^{18},q^{26};q^{32}\right]_\infty}{\left[q^7;q^{16}\right]_\infty\left[q^2,q^6,q^8,q^{16},q^{22};q^{32}\right]_\infty}$$

(4.8)
$$-\frac{2\left[q^4,q^{17};q^{16}\right]_\infty\left[q^2,q^4,q^{10},q^{18};q^{32}\right]_\infty}{q^2\left[q^7;q^{16}\right]_\infty\left[q^2,q^6,q^8,q^{16},q^{22},q^{36};q^{32}\right]_\infty},$$

(4.9)
$$\beta_2\left(q^2\right)=\frac{1}{2}[S(q)^{-1}-S(-q)^{-1}].$$

By adding (4.7) and (4.9), we get (3.2)

$$S(q)^{-1}=\alpha_2\left(q^2\right)+\beta_2\left(q^2\right)$$

$$S(q)^{-1}=\frac{\left[q^2,q^{10};q^{16}\right]_\infty\left[q^{18};q^{32}\right]_\infty}{\left[q^8;q^{16}\right]_\infty\left[q^4,q^{16},q^{22};q^{32}\right]_\infty}+\frac{q^2\left[q^2,q^6;q^{16}\right]_\infty\left[q^2;q^{32}\right]_\infty}{\left[q^8;q^{16}\right]_\infty\left[q^4,q^6,q^{16};q^{32}\right]_\infty}$$
$$-\frac{q^2\left[q,q^4;q^{16}\right]_\infty\left[q^{18};q^{32}\right]_\infty}{\left[q^7;q^{16}\right]_\infty\left[q^8,q^{16},q^{22};q^{32}\right]_\infty}-\frac{\left[q^4,q^{17};q^{16}\right]_\infty\left[q^4,q^{18};q^{32}\right]_\infty}{q^2\left[q^7;q^{16}\right]_\infty\left[q^6,q^8,q^{16},q^{36};q^{32}\right]_\infty}.$$

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