ISSN 0974 - 9373

Vol. 15 No.2 (2011) **Journal of International Academy of Physical Sciences** pp. 175-180

Some Fixed Point Theorems in G-Metric Space

Rajendra Pant
Department of Mathematics, Walter Sisulu University
Mthatha 5117, South Africa
E-mail: pant.rajendra@gmail.com

Rakesh Mohan
Department of Mathematics, Dehradun Institute of Technology
Mussourie-Diversion Road, Makkawala, Dehradun, (Uttarakhand) 248009, India
Email: rkmohan2k4@yahoo.com

Pankaj Kumar Mishra
Department of Mathematics, University of Petroleum & Energy Studies
P.O. Bidholi, Via Prem Nagar, Dehradun (Uttarakhand) 248007, India
Email: pk_mishra009@yahoo.co.in

(Received December 22, 2010)

Abstract: In this note we obtain some fixed point theorems on G-metric space. Our result is an extension of a recent fixed point theorems of Shatanawi [1].

Keywords: G-metric space, fixed point, (IT)-commuting.
2000 AMS Classification No.: 54H25, 47H10.

1. Introduction

Metric fixed point theory is an important mathematical discipline because of its applications in areas such as variational and linear inequalities, optimization, and approximation theory. The fixed point theorems in metric spaces are playing a major role to construct methods in mathematics to solve problems in applied mathematics and sciences. So the attraction of metric spaces to a large numbers of mathematicians is understandable. Some generalizations of the notion of a metric space have been proposed by some authors. To overcome fundamental flaws in Dhage's theory of Generalized metric spaces[2], Mustafa and Sims[3] introduced a more appropriate generalization of metric spaces, that of G-metric spaces. Afterwards, Mustafa et. Al[4] obtained several fixed point theorems for mappings satisfying different contractive conditions in G-metric spaces. In fact, Mustafa, Braliy and others studied many fixed point results for a self mapping in G-metric space under certain conditions see [5-7]. In this paper, we study some fixed point results for self mapping in a complete G-metric space X under some contractive conditions related to a non-decreasing map $\varphi:[0,+\infty)\to[0,+\infty)$ with $\lim_{n\to\infty}\varphi^n(t)=0$ for all $t\in(0,+\infty)$.

2. Preliminaries

We recall some definitions and properties for G-metric spaces given by Mustafa and Sims.

Definition 2.1.[3] Let X be a nonempty set, and let $G: X\times X\times X\to R^+$ be a function satisfying the following axioms:

(G1) $G(x,y,z)=0$ if $x=y=z$,

(G2) $0<G(x,x,y)$ for all $x,y\in X$ with $x\neq y$,

(G3) $G(x,x,y)\leq G(x,y,z)$ for all $x,y,z\in X$ with $z\neq y$,

(G4) $G(x,y,z)=G(x,z,y)=G(y,z,x)=\ldots,$, symmetry in all three variables

(G5) $G(x,y,z)\leq G(x,a,a)+G(a,y,z)$ for all $x,y,z,a\in X$.

Then the function G is called a generalized metric, or, more specifically, a G-metric on X , and the pair (X,G) is called a G-metric space.

Definition 2.2.[3] Let (X,G) be a G-metric space, and let $\{x_n\}$ a sequence of points in X , a point 'x' in X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\to\infty}G(x,x_n,x_m)=0,$

and one says that sequence $\{x_n\}$ is G-convergent to x.

Proposition 1.[3] *Let (X,G) be a G-metric space. Then the following are equivalent:*

(i) *$\{x_n\}$ is G-convergent to x,*

(ii) $G(x_n,x_n,x)\to 0$ *as* $n\to\infty$,

(iii) $G(x_n,x,x)=0$ *as* $n\to\infty$,

(iv) $G(x_m,x_n,x)=0$ *as* $m,n\to\infty$,

Definition 2.3.[3] Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if, for each $\varepsilon>0$, there exist a positive integer N such that $G(x_m,x_n,x_l)<\varepsilon,$ for all $n,m,l\geq N$.

Proposition 2.[3] *Let (X,G) be a G-metric space. Then the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

Definition 2.4. [3] A G-metric space (X,G) is said to be G-complete if every G-Cauchy Sequence in (X,G) is G-convergent in X.

Proposition 3. [3] *Let (X,G) be a G-metric space. Then for $x,y,z\in X$ it follows that:*

(i) *If $G(x,y,z)=0$ then $x=y=z$,*

(ii) $G(x,y,z)\le G(x,x,y)+G(x,x,z)$,

(iii) $G(x,y,y)\le 2G(y,x,x)$,

(iv) $G(x,y,z)\le G(x,a,z)+G(a,y,z)$,

(v) $G(x,y,z)\le \frac{2}{3}(G(x,y,a)+G(x,a,z)+G(a,y,z))$,

(vi) $G(x,y,z)\le (G(x,a,a)+G(y,a,a)+G(z,a,a))$.

Example [3] Let (R,d) be the usual metric space. Define G_s by

$$G_s(x,y,z)=d(x,y)+d(y,z)+d(x,z),$$

for all $x,y,z\in R$. Then it is clear that (R,G_s) is a G-metric space.

Definition 2.5. Let $f,g:X\to X$ Then the pair (f,g) is said to be (IT)-commuting at $z\in X$, if $fgz=gfz$.

3. Main Result

Following Matkowski [8], let Φ be the set of all functions φ such that $\varphi:[0,+\infty)\to[0,+\infty)$ be a non-decreasing function with $\lim_{n\to+\infty}\varphi^n(t)=0$ for all $t\in(0,+\infty)$. If $\varphi\in\Phi$, then φ is called Φ-map. If φ is Φ-map, then it is easy to show that:

1. $\varphi(t)<t$ for all $t\in(0,+\infty)$.
2. $\varphi(0)=0$.

From now unless otherwise stated we mean by φ is Φ-map. Now, we introduce and prove our first result.

Theorem 3.1. *Let (X,G) be a complete G-metric space, suppose $f,g:X\to X$ be a Φ-pair, there exist a φ-map satisfy,*

$$G(fx,fy,fz)\le\varphi(G(gx,gy,gz)) \text{ for all } x,y,z\in X. \quad (3.1)$$

Suppose that f *and* g *are* (IT)-*commuting with* $f(X) \subset g(X)$. *If* $f(X)$ *or* $g(X)$ *is a complete subspace of* X, *then the mappings* f *and* g *have a unique common fixed point in* X. *Moreover for any* $x_0 \in X$, *every* f - g - *sequence* $\{fx_n\}$ *with initial point* x_0 *converges to the common fixed point.*

Proof: Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done, since the range of g contains the range of f. Continuing this process, having chosen $x_n \in X$, we obtain $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$. Then

$$G(fx_{n+1}, fx_n, fx_n) \le \varphi(G(gx_{n+1}, gx_n, gx_n)) = \varphi(G(fx_n, fx_{n-1}, fx_{n-1})).$$

Consequently $G(fx_{n+1}, fx_n, fx_n) \le \varphi^n(G(fx_1, fx_0, fx_0))$.
Given $\varepsilon > 0$, since $\lim_{n\to\infty} \varphi^n(G(fx_1, fx_0, fx_0)) = 0$ and $\varphi(\varepsilon) < \varepsilon$,
there is an integer δ such that

$$\varphi^n(G(fx_1, fx_0, fx_0)) < \varepsilon - \varphi(\varepsilon) \text{ for all } n \ge \delta.$$

Hence

$$G(fx_{n+1}, fx_n, fx_n) \le \varepsilon - \varphi(\varepsilon) \text{ for all } n \ge \delta. \tag{3.2}$$

For $m, n \in N$ with $m > n$, we claim:

$$G(fx_m, fx_n, fx_n) < \varepsilon \text{ for all } m \ge n \ge \delta. \tag{3.3}$$

We prove Inequality (3.3) by induction on m. Inequality (3.3) holds for $m = n+1$ by using Inequality (3.2). Assume Inequality (3.3) holds for $m = k$.
For $m = k+1$, we have

$$\begin{aligned} G(fx_{k+1}, fx_n, fx_n) &\le G(fx_{n+1}, fx_n, fx_n) + G(fx_{k+1}, fx_{n+1}, fx_{n+1}), \\ &< \varepsilon - \varphi(\varepsilon) + \varphi(G(gx_{k+1}, gx_{n+1}, gx_{n+1})), \\ &< \varepsilon - \varphi(\varepsilon) + \varphi(G(fx_k, gx_n, gx_n)), \\ &< \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) = \varepsilon. \end{aligned}$$

By induction on m, we conclude that Inequality (3.3) holds for all $m \ge n \ge \delta$. So $\{fx_n\}$ is G-Cauchy sequence. Suppose that $f(X)$ is a complete subspace of X, then there exists $y \in f(X) \subset g(X)$ such that

$f(x_n) \to y$ and also $g(x_n) \to y$. (This holds also if $g(X)$ is complete with $y \in g(X)$.) Let $z \in X$ be such that $g(z) = y$. We show that $fz = gz$.
Fix $\varepsilon \gg 0$ and we choose a natural number N such that $G(y, y, fx_n) \ll \varepsilon/2$ and $G(gx_n, gx_n, gz) \ll \varepsilon/2$.

Then

$$\begin{aligned} G(y, y, fz) &\leq G(y, y, fx_n) + G(fx_n, fx_n, fz), \\ &\leq G(y, y, fx_n) + \varphi(G(gx_n, gx_n, gz)), \\ &< G(y, y, fx_n) + G(gx_n, gx_n, gz), \\ &\ll \varepsilon / 2 + \varepsilon / 2 = \varepsilon, \end{aligned}$$

Thus, $G(y, y, fz) \ll \varepsilon/m$ for all natural number m.
From $\varepsilon/m - G(y, y, f(z)) \in X$, for all m, as $m \to +\infty$ we obtain $G(y, y, fz) \in X$.
Therefore $G(y, y, fz)) = 0$ which implies $y = fz = gz$, that is z is a coincidence point of f and g and y a point of coincidence of f and g.

Now, we use the hypothesis that f and g are (IT)-commuting to deduce that y is a common fixed point. From $fz = gz$, by the definition (2.5), it follows that

$$fy = fgz = ggz = gy,$$

We show that $f(y) = g(y) = y$. If $g(y) \neq y$, in virtue of (3.1), we obtain

$$G(fy, fy, fz) \leq \varphi(G(gy, gy, gz)) < G(gy, gy, gz) = G(fy, fy, fz),$$

which gives $f(y) = y = g(y)$. Then y is a common fixed point for the mappings f and g. The uniqueness follows from the hypothesis that f and g are a Φ-pair.

From the proof of Theorem 3.1 we deduce the following result on points of coincidence.

Corollary 3.1. *Let* (X, G) *be a* G*-metric space, and* $f, g : X \in X$ *be a* Φ*-pair. Suppose that* $f(X) \subset g(X)$. *If* $f(X)$ *or* $g(X)$ *is a complete subspace of* X, *then the mappings* f *and* g *have a unique point of coincidence in* X. *Moreover for any* $x_0 \in X$, *every* f - g *-sequence* $\{f(x_n)\}$ *with initial point* x_0 *converges to the point of coincidence.*

If in Theorem 3.1 we choose the Φ-map defined by $\varphi(\omega) = k\omega$, where $k \in [0,1)$ is a constant, we obtain the following theorem.

Theorem 3.2. *Let (X,G) be a G-metric space, suppose $f,g: X \in X$ satisfies*

$$G(fx, fy, fz) \leq k\, G(gx, gy, gz) \text{ for every } x, y, z \in X,$$

where $k \in [0,1)$ is a constant. If $f(X) \subset g(X)$, and $f(X)$ or $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover if f and g are (IT)-*commuting then f and g have a unique common fixed point.*

Acknowledgement

The authors thank Prof. S. L. Singh for his valuable suggestions and encouragement.

References

1. W. Shatanawi, Fixed point theory for contractive mappings satisfying ϕ-maps in G-metric spaces, *Fixed Point Theory and Applications*, **9** (2010).
2. B. C. Dhage, Generalized metric space and mapping with fixed point, *Bulletin of Calcutta Mathematical Society*, **84** (1992) 329-336.
3. Z. Mustafa and B. Sims, A new approach to generalized metric spaces, *Journal of Nonlinear and Convex Analysis,* **7** (2006) 289-297.
4. O. Hamed Z. Mustafa and F. Awawdeh, Some fixed point theorem for mappings on complete G-metric spaces, *Applied General Topology,* **12** (2008).
5. Z. Mustafa and B. Sims, Some remarks concerning D-metric spaces, *International Conference on Fixed Point Theory and Applications*, Yokohama, Japan, **10** (2004).
6. Z. Mustafa and B. Sims, Fixed point theorems for contractive mappings in complete G-metric spaces, *Fixed Point Theory and Applications,* **10** (2009).
7. W. Shatanawi Z. Mustafa and M. Bataineh, Existence of fixed point results in G-metric spaces, *International Journal of Mathematics and Mathematical sciences,* (2009) 10.
8. J. Matkowski, Existence of fixed point results in G-metric spaces, *Proceedings of American Mathematical Society*, **62** (1977) 344-348.