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Some Study on the Theory of Fixed Points in a Metric Space

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Abstract: The present paper deals with some results on fixed point theory in metric space and analyzes the analogous concepts and the related results under a generalized mapping condition, supported by an example. The work extends the results of Taskovic[1] and Som[2, 3].

Key words: Metric space, Weakly commuting mappings, Generalized mapping condition, Common fixed point.
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1. Introduction

Let F be a mapping of a set X into itself. An element $x \in X$ is said to be a fixed point of the mapping F if $Fx = x$. By a fixed point theorem we shall understand a statement which asserts that under certain conditions (on the mapping F and on the space X) a mapping F of X into itself admits one or more fixed points. The Banach contraction theorem assures fixed point for a mapping which is necessarily continuous. However Taskovic[1] proved the following fixed point result for a self-mapping which is not necessarily continuous.

Theorem1.1. *Let T be a self-mapping of a complete metric space (X, d) satisfying*

(1.1) $$ad(Tx,Ty) + bd(x,Tx) + cd(y,Ty) - min\{d(Tx, y), d(x,Ty)\} \le qd(x, y),$$

for all $x, y \in X$, where $a,b,c \ge 0$ with $a > q+1$ and $a + c > 0$.
Then T has a unique fixed point in X.

Generalizing the above result of Taskovic for common fixed point of two mappings, Som[2] obtained the following results.

Theorem 1.2. *Let T and S be two self-mappings of a complete metric space $(X,\ d)$ satisfying*

$$(1.2)\quad ad(Tx,Sy)+bd(x,Tx)+cd(y,Sy)-min\{d(x,Sy),d(Tx,y)\}\le qd(x,y),$$

for all $x,y\in X$, where $a,b,c\ge 0$, $q>0$ with $a>q+1$ and $a+c>0$.
Then T and S have a unique common fixed point.

Theorem 1.3. *Let $(X,\ d)$ be a complete metric space and T and S be self mappings of X satisfying*

$$(1.3)\quad \begin{aligned}&ad(Tx,Sy)+bd(x,Tx)+cd(y,Sy)-min\{d(x,Sy),d(Tx,y)\}\\&\le q\,max\{d(x,y),d(x,Tx),d(y,Sy),[d(x,Sy)+d(Tx,y)]/2\},\end{aligned}$$

for all $x,y\in X$, where $a,b,c\ge 0$, $q>0$ with $a>q+1$ and $a+c>0$.
Then T and S have a unique common fixed point.

Unifying the mapping conditions of Theorems 1.1 and 1.2 we obtain few common fixed point results in the next section.

2. Some Results on Metric Space

Our first common fixed point result on metric space goes as follows:

Theorem 2.1. *Let $(X,\ d)$ be a complete metric space and T, S be self-mappings of X satisfying*

$$(2.1)\quad \begin{aligned}&ad(Tx,Sy)+bd(x,Tx)+cd(y,Sy)\\&\le q\,max\{d(x,y),d(x,Tx),d(y,Sy),\alpha[d(x,Sy)+d(Tx,y)]\},\end{aligned}$$

for all $x,y\in X$, where $a,b,c\ge 0$, $q>0$, $\alpha<1/2$ with $a>q$.
Then T and S have a unique common fixed point in X.

Proof: Let $x_0\in X$ be any arbitrary point. Define a sequence

$\{x_n\}$ recursively as $x_1=Tx_0, x_2=Sx_1,...,x_{2n-1}=Tx_{2n-2}, x_{2n}=Sx_{2n-1}$.
Let $d_n=d(x_n,\ x_{n+1})>0$ for all $n=0,1,2,....$

From (2.1) we get by putting $x = x_{2n\text{-}2},\ y = x_{2n\text{-}1}$

$$ad(Tx_{2n\text{-}2},\ Sx_{2n\text{-}1}) + bd(x_{2n\text{-}2},\ Tx_{2n\text{-}2}) + cd(x_{2n\text{-}1},\ Sx_{2n\text{-}1})$$
$$\leq q\ \max\{d(x_{2n\text{-}2}, x_{2n\text{-}1}), d(x_{2n\text{-}2},\ Tx_{2n\text{-}2}),\ d(x_{2n\text{-}1},\ Sx_{2n\text{-}1}),$$
$$\alpha[d\ (x_{2n\text{-}2},\ Sx_{2n\text{-}1}) + d\ (Tx_{2n\text{-}2},\ x_{2n\text{-}1})]\}$$

or

$$ad(x_{2n\text{-}1},\ x_{2n}) + bd(x_{2n\text{-}2},\ x_{2n\text{-}1}) + cd(x_{2n\text{-}1},\ x_{2n})$$
$$\leq q\ \max\ \{d(x_{2n\text{-}2},\ x_{2n\text{-}1}),\ d(x_{2n\text{-}2},\ x_{2n\text{-}1}), d(x_{2n\text{-}1},\ x_{2n}),$$
$$\alpha\ [d(x_{2n\text{-}2},\ x_{2n}) + d(x_{2n\text{-}1},\ x_{2n\text{-}1})]\}$$

or

$$ad_{2n-1} + bd_{2n-2} + cd_{2n-1} \tag{2.2}$$
$$\leq q\ \max\ \left\{d_{2n-2}, d_{2n-2}, d_{2n-1}, \alpha\left[d_{2n-2} + d_{2n-1}\right]\right\}.$$

Now putting $k = \max\ \left\{d_{2n-2}, d_{2n-1}\right\}$ we get $d_{2n-2} \leq k$ and $d_{2n-1} \leq k$ implying that

$$\alpha\left(d_{2n-2} + d_{2n-1}\right)/2 \leq \alpha\, k.$$

Thus from (2.2) we get

$$\left(a+c\right) d_{2n-1} + bd_{2n-2} \leq q\ \max\ \left\{d_{2n-1}, d_{2n-2},\ 2\alpha k\right\}. \tag{2.3}$$

Case I. Let $d_{2n-2} \leq d_{2n-1}$, i.e., $k = d_{2n-1}$, then (2.3) implies that

$$\left(a + c\right) d_{2n-1} \leq -b\, d_{2n-2} + q\, d_{2n-1}$$

$$\Rightarrow d_{2n-1} \leq \frac{-b}{a+c-q} d_{2n-2} = p^{/} d_{2n-2}, \quad \text{where } p^{/} = \frac{-b}{a+c-q} < 1,$$

(for $a > q \Rightarrow a + b + c > q$).

Case II. If $d_{2n-1} \leq d_{2n-2}$ then (2.3) implies that

$$\left(a + c\right) d_{2n-1} + bd_{2n-2} \leq q\, d_{2n-2}$$

$$\Rightarrow \qquad d_{2n-1} \leq \frac{q-b}{a+c} d_{2n-2} = p^{//} d_{2n-1}, \text{ where } p^{//} = \frac{q-b}{a+c} < 1$$

Therefore $d_{2n-1} \leq p\ d_{2n-2} \leq \ \ldots\ldots.. \leq p^{2n-1} d_o \to 0$ as $n \to \infty$, where $p = \max\{p^{/},\ p^{//}\}$.

Thus in both the cases $\{x_n\}$ is a Cauchy sequence in X .Since X is complete $\{x_n\}$ converges to some point $u \in X$. Clearly the subsequences $\{Tx_{2n-2}\}$ and $\{Sx_{2n-1}\}$ also converge to u. Now in (2.1) putting $x = x_{2n-2}$ and $y = u$ we get

$$\begin{aligned} &ad(Tx_{2n-2}, Su) + bd(x_{2n-2}, Tx_{2n-2}) + cd(u, Su) \\ &\le q \max\{d(x_{2n-2}, u), d(x_{2n-2}, Tx_{2n-2}), d(u, Su), \\ &\qquad \alpha[d(x_{2n-2}, Su) + d(Tx_{2n-2}, u)], \end{aligned}$$

which in the limiting case gives

$$(a+c-q)\, d(u, Su) \le 0.$$

Therefore, $Su = u$ (for $a+c > q$). Thus u is a fixed point of S.
Similarly by putting $x = u$, $y = x_{2n-1}$ in (2.1) it can be shown that u is a fixed point of T, proving also that u is a common fixed point of T and S.
To prove uniqueness let $v\ (\neq u)$ be another common fixed point of T and S. Then from (2.1) we get

$$ad(Tu, Sv) + bd(u, Tu) + cd(v, Sv)$$
$$\le q \max \left\{ \begin{array}{l} d(u,v), d(u,Tu), d(v,Sv), \\ \alpha\left[d(u,Sv] + d(Tu,v)\right] \end{array} \right\},$$

$$\Rightarrow \quad (a-q)\, d(u,v) \le 0$$
$$\Rightarrow \quad u = v \quad for\, a - q > 0.$$

Thus u is the unique common fixed point of T and S. This completes the proof of the theorem.

The following example shows that the above theorem is a generalization of Theorem 1.2 Som[2].

Example 2.1. Let $X = [0, 1]$ and $T, S : X \to X$ be such that

$$T(x) = \begin{cases} 2x/3 & 0 \le x \le ½ \\ x^2/4 & ½ < x \le 1 \end{cases} \qquad S(y) = \begin{cases} y^2/3 & 0 \le y < 1 \\ y/2 & y = 1. \end{cases}$$

Let $d(x, y) = |x - y|$ for all $x, y \in X$ be the usual metric. Then clearly T and S are not continuous at $x = ½$ and $y = 1$ respectively.

Now at $x = 1$ and $y = 1/2$, the inequality (2.1) leads to

$$ad\left(T(1),\ S(½)\right) + bd\left(1,\ T(1)\right) + cd\left(½,\ S(½)\right)$$

$$\leq q \max \left\{ \begin{array}{l} d((1,\ ½),\ d\left(1,\ T(1)\right),\ d\left(½,\ S(½)\right), \\ \alpha\, [d(1,\ S(½) + d\ \left(T(1),\ ½\right)] \end{array} \right\}.$$

$$\Rightarrow \quad ad\ \left(1/4, 1/12\right) + bd\ \left(1, 1/4\right) + cd\ \left(1/2, 1/12\right)$$

$$\leq q \max \left\{ \begin{array}{l} d\left(1, 1/2\right),\ d\ \left(1, 1/4\right),\ d\ \left(1/2, 1/12\right), \\ \alpha \left[d\left(1, 1/12\right) + d\ \left(1/4, 1/2\right)\right] \end{array} \right\}$$

$$\Rightarrow \quad \frac{1}{6} a + \frac{1}{6} b + \frac{1}{2} c \leq q \max \left\{1/2,\ 1/6,\ 1/2, (2/3)\alpha\right\}$$

$$\Rightarrow \quad (1/6)a + (3/4)b + (5/12)c \leq q \max\{1/2, 3/4, 5/12, (7/6)\alpha\}$$

$$= (3/4)q \qquad \forall\ \alpha < 1/2$$

Taking $a = 73/12$, $b = 1/9$, $c = 2/3$ and $q = 2$ we get $11/8 \leq 3/2$, which is true. Thus the example satisfies the inequality (2.1) of Theorem 2.1. However putting all these values in inequality (1.2) we get $11/8$ not $\leq 5/4$, i.e., these values does not satisfy inequality (1.2) and 0 is the unique common fixed point of T and S.

Our next result on common fixed point of four mappings goes as follows:

Theorem 2.2. *Let $(X,\ d)$ be a complete metric space. Let T, S, H and G be self mappings of X with $T(X) \subseteq H(X)$ and $S(X) \subseteq G(X)$ and satisfy*

(2.4) $ad(Tx,\ Sy)d(Gx,\ Hy) + bd(Tx,\ Gy)d(Hy,\ Sy) + cd(Sy,\ Hy)d(Gx,\ Hy)$

$$\leq q \max\ \{d(Tx,\ Gx)d(Hy,\ Gx),\ d(Tx,\ Gx)d(Sy,\ Hy),\ d(Hy,\ Gx)(Sy,\ Tx)\}$$

for all $x, y \in X$ where $a, b, c, q \geq 0$ with $a > q$. Then T, S, H and G have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be any arbitrary point. As $T(X) \subseteq H(X)$ and $S(X) \subseteq G(X)$, so we get a sequence $\{y_n\}$ in X defined as $Tx_0 = Hx_1 = y_1$ (say), $Sx_1 = Gx_2 = y_2$ (say)… . In general

$$Tx_{2n} = Hx_{2n+1} = y_{2n+1}, Sx_{2n+1} = Gx_{2n+2} = y_{2n+2}, \ n = 0,1,2, \ \ldots .$$

Now putting $x = x_{2n-2}$, $y = x_{2n-1}$ in (2.4) we get

$$\begin{aligned}\{ad(Tx_{2n-2}, Sx_{2n-1})d(Gx_{2n-2}, Hx_{2n-1}) &+ bd(Tx_{2n-2}, Gx_{2n-2})\, d(Hx_{2n-1}, Sx_{2n-1}) \\ &+ cd(Sx_{2n-1}, Hx_{2n-1})d(Gx_{2n-2}, Hx_{2n-1})\ \} \\ &\le q \ \max \begin{Bmatrix} d(T\,x_{2n-2}, G\,x_{2n-2})d(Hx_{2n-1}, Gx_{2n-2}), \\ d(Tx_{2n-2}, Gx_{2n-2})d(Sx_{2n-1}, Hx_{2n-1}), \\ d(Hx_{2n-1}, Gx_{2n-2})(Sx_{2n-1}, Tx_{2n-2}) \end{Bmatrix},\end{aligned}$$

i.e.

$$\begin{aligned}ad(y_{2n-1}, y_{2n})d(y_{2n-2}, y_{2n-1}) &+ bd(y_{2n-1}, y_{2n-2})d(y_{2n-1}, y_{2n}) \\ &+ cd(y_{2n}, y_{2n-1})d(y_{2n-2}, y_{2n-1}) \\ &\le q \ \max \begin{Bmatrix} d(y_{2n-1}, y_{2n-2})d(y_{2n-1}, y_{2n-2}), \\ d(y_{2n-1}, y_{2n-2})d(y_{2n}, y_{2n-1}), \\ d(y_{2n-1}, y_{2n-2})d(y_{2n}, y_{2n-1}) \end{Bmatrix}.\end{aligned}$$

Now let $d_n = d(y_n, y_{n-1})$, $n = 2, 3, \ldots$. Then we get

i.e.;
$$\begin{aligned}&ad_{2n}d_{2n-1} + bd_{2n-1}d_{2n} + cd_{2n}d_{2n-1} \\ &\le q \ \max \ \{d_{2n-1}d_{2n-1}, d_{2n-1}d_{2n}, d_{2n-1}d_{2n}\},\end{aligned}$$

(2.5) $$(a+b+c)d_{2n-1}d_{2n} \le q\, d_{2n-1} \max \ \{ d_{2n-1}, d_{2n}\}.$$

Now consider $d_{2n-1} < d_{2n}$ (for large n) then (2.5) gives

$$(a+b+c)\, d_{2n} \le q\, d_{2n} \quad \text{i.e.,} \quad (a+b+c-q)d_{2n} \le 0,$$

which is a contradiction since $a+b+c > q$.

So this case cannot be accepted. Therefore only possibility is that $d_{2n} < d_{2n-1}$, then (2.5) gives

$(a+b+c)d_{2n} \le qd_{2n-1}$ i.e. $d_{2n} \le p\, d_{2n-1}$, where $p = q/(a+b+c) < 1.$

As such $d_{2n} \le pd_{2n-1} \le p^2 d_{2n-2} \le \ \ldots \ \le p^{2n-2} d_2 \to 0$ as $n \to 0$.

Thus $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, $\{y_n\}$ converges to some point $u \in X$. Consequently the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ also converge to the same limit u.

Thus $\{Sx_{2n-1}\}$, $\{Tx_{2n}\}$ and consequently $\{Gx_{2n}\}$, $\{Hx_{2n-1}\}$ also converge to u. Now taking $x = x_{2n-2}$, $y = u$ in (2.4), we get

$$\begin{aligned} &ad(Tx_{2n-2}, Su)\, d(Gx_{2n-2}, Hu) + bd(Tx_{2n-2}, Gx_{2n-2})\, d(Hu, Su) \\ &\qquad + cd(Su, Hu)\, d(Gx_{2n-2}, Hu) \\ &\qquad \le q \max \begin{Bmatrix} d(Tx_{2n-2}, Gx_{2n-2})\, d(Hu, Gx_{2n-2}), \\ d(Tx_{2n-2}, Gx_{2n-2})\, d(Su, Hu), \\ d(Hu, Gx_{2n-2})(Su, Tx_{2n-2}) \end{Bmatrix}, \end{aligned}$$

i.e.,

$$\begin{aligned} &ad(y_{2n-1}, Su)\, d(y_{2n-2}, Hu) + bd(y_{2n-1}, y_{2n-2})\, d(Hu, Su) \\ &\qquad + cd(Su, Hu)\, d(y_{2n-2}, Hu) \\ &\qquad \le q \max \begin{Bmatrix} d(y_{2n-1}, y_{2n-2})\, d(Hu, y_{2n-2}), \\ d(y_{2n-1}, y_{2n-2})\, d(Su, Hu), \\ d(Hu, y_{2n-2})\, d(Su, y_{2n-1}) \end{Bmatrix}. \end{aligned}$$

In the limiting case we get

$$ad(u, Su)\, d(u, Hu) + cd(Su, Hu)\, d(u, Hu) \le qd(Hu, u)\, d(Su, u)$$

i.e.,

$$\{(a-q)\, d(u, Su) + cd(Su, Hu)\}\, d(Hu, u) \le 0.$$

So there are two cases as discussed below:

Case 1: $$(a-q)\, d(u, Su) + cd(Su, Hu) \le 0$$

but both the quantities on left hand side of the above equation are nonnegative, so both must be identically equal to zero i.e.,

$$d(Su, u) = 0 \text{ and } d(Su, Hu) = 0 \quad \text{i.e.,} \quad Su = Hu = u.$$

Case 2: $d(Hu,\ u) = 0$, i.e $Hu = u$.

Thus u is a fixed point of H or S and H both. By putting $x = u$ and $y = x_{2n-1}$ in (2.4) we get

$$ad(Tu,\ Sx_{2n-1})\,d(Gu,\ Hx_{2n-1}) + bd(Tu,\ Gx_{2n-2})\ d(Hx_{2n-1},\ Sx_{2n-1})$$
$$+ cd(Sx_{2n-1},\ Hx_{2n-1})\,d(Gu,\ Hx_{2n-1})$$
$$\leq q \max\ \{d(Tu,\ Gu)\,d(Hx_{2n-1},\ Gu),\ d(Tu,Gu)\,d(Sx_{2n-1},\ Hx_{2n-1}),$$
$$d(Hx_{2n-1},\ Gu)(Sx_{2n-1},\ Tu)\}$$

i.e.

$$ad(Tu,\ y_{2n})\,d(Gu,\ y_{2n-1}) + bd(Tu,\ y_{2n-2})\,d(y_{2n-1},\ y_{2n})$$
$$+ cd(y_{2n},\ y_{2n-1})\,d(Gu,\ y_{2n-1})$$
$$\leq q \max \begin{Bmatrix} d(Tu,\ Gu)\,d(y_{2n-1},\ Gu),\ d(Tu,\ Gu)\,d(y_{2n},\ y_{2n-1}), \\ d(y_{2n-1},\ Gu)\,d(y_{2n},\ Tu) \end{Bmatrix}.$$

In the limiting case as $n \to \infty$ we get

$$ad(Tu,\ u)\ d(Gu,\ u) + bd(Tu,\ u)\ d(u,\ u) + cd(u,\ u)\ d(Gu,\ u)$$
$$\leq q \max \begin{Bmatrix} d(Tu,\ Gu)\ d(u,\ Gu),\ d(Tu,\ Gu)\ d(u,\ u), \\ d(u,\ Gu)\ d(u,\ Tu) \end{Bmatrix},$$

$$ad(Tu,u)\,d(Gu,u) \leq q \max\{d(Tu,Gu)\ d(u,Gu), d(u,Gu)\,d(u,Tu)\},$$

(2.6) $ad(Tu,\ u)\,d(Gu,\ u) \leq q\ d(u,\ Gu) \max\{d(Tu,\ Gu),\ d(u,\ Tu)\}$.

Now if (a) $d(Tu,\ Gu) \leq d(u,\ Tu)$ then (2.6) leads to

$$(a-q)\ d(Tu,\ u)\ d(Gu,\ u) \leq 0\ ,$$

implying that either $Tu = u$ or $Gu = u$ or both $Tu = u$ and $Gu = u$. Accordingly the mappings S, H, T or S, H, G or S, H, T, G have a common fixed point in case 1 respectively and H, T or G, H or H, T, G have a common fixed point in case 2 respectively.

Next if (b) $d(u,\ Tu) \leq d(Tu,\ Gu)$ then by (2.6) we have

$$ad(Tu,\ u)\ d(Gu,\ u)\ \le\ q\ d(u,\ Gu)\ d(Tu,\ Gu)$$

or, $$d(Gu,\ u)\ \{ad(Tu,\ u) - q\ d(Tu,\ Gu)\} \le 0$$

implying that either $Gu = u$ or $ad(Tu,\ u) \le q\,d(Tu,\ Gu)$ i.e. $Tu = u$ if $Tu = Gu$ as such S, H, G have a common fixed point in case 1 and S, H, T, G have a common fixed point in case 2 provided $Tu = Gu$. Uniqueness of the fixed point can be easily proved using (2.4). So we omit it.

3. A Common Fixed Point Result for Weakly Commuting Mappings

In 1982 Sessa[4] introduced the notion of weak commutativity and showed that a commuting pair of mappings is always weakly commuting but not the converse. In this section we obtain a common fixed point result for such mappings generalizing an earlier result of Som[2] in respect of the mapping structure as well as the mapping condition and also the results of Jungck[1] and Taskovic[1] .

Definition 3.1. Two self mappings A or S of a metric space are called weakly commuting if

$$d(ASx,\ SAx) \le d(Ax,\ Sx) \qquad \text{for all } x \in X\ .$$

Theorem 3.1. *Let $(X,\ d)$ be a complete metric space. Let f and g be a pair of weakly commuting self mappings of X with f continuous and $g(X) \subseteq f(X)$. Let g satisfy*

$$\begin{aligned} &a_1 d(gx,\ gy)\ +\ a_2 d(fx,\ gx)\ +\ a_3 d(fy,\ gy)\ +\ a_4 d(fx,\ fy) \\ &\quad \le\ q\ \max\{d(fx,\ fy), d(fx,\ gy), d(gx,\ fy), d(gx,\ gy)\} \end{aligned} \tag{3.1}$$

for all $x,\ y \in X$ with $a_i \ge\ 0, q\ >\ 0$ with $a_1 + a_3 > q, a_1 + a_4 > q$ and $q < (a_1 + a_2 + a_3 + a_4)/2$. Then f and g have a unique common fixed point in X .

Proof : Let x_0 be any arbitrary point in X . Since $g(X) \subseteq f(X)$ let $x_0 \in X$ then there is some $x_1 \in X$ such that $g(x_0) = f(x_1) = y_1$ (say). For this x_1, there is some $x_2 \in X$, such that $g(x_1) = f(x_2) = y_2$ (say) and so on. In general we get points x_{n-1} and $x_n \in X$ such that

$$g\left(x_{n-1}\right)=f\left(x_n\right)=y_n,\ \ n\ =\ 1,\ 2,\ldots..\ .$$

Let $d_n\ =\ d\left(y_n,\ y_{n+1}\right), n\ =\ 1,\ 2,\ldots..$ Then $d_n \geq 0.$
Now putting $x\ =\ x_{n+1},\ y\ =\ x_n$ in (3.1) we get

$$\begin{aligned} & a_1 d\left(gx_{n+1},\ gx_n\right)\ +\ a_2 d\left(fx_{n+1},\ gx_{n+1}\right)\ +\ a_3 d\left(fx_n,\ gx_n\right)\ +\ a_4 d\left(fx_{n+1},\ fx_n\right) \\ & \leq\ q\ \max\left\{d\left(fx_{n+1},\ fx_n\right),\ d\left(fx_{n+1},\ gx_n\right),\ d\left(gx_{n+1},\ fx_n\right),\ d\left(gx_{n+1},\ gx_n\right)\right\} \\ & \leq\ q\ \max\left\{d\left(y_{n+1},\ y_n\right),\ d\left(y_{n+1},\ y_{n+1}\right),\ d\left(y_{n+2},\ y_n\right),\ d\left(y_{n+2},\ y_{n+1}\right)\right\} \end{aligned}$$

or $$\left(a_1+\ a_2\right)d_{n+1}+\ \left(a_3+\ a_4\right)d_n \leq\ q\ \max\left\{d_n,\ 0,\ d_n+\ d_{n+1},\ d_{n+1}\right\}$$

or $$\left(a_1+\ a_2\right)d_{n+1}+\ \left(a_3+\ a_4\right)d_n \leq\ q\left(d_n+\ d_{n+1}\right)$$

or $$\left(a_1+\ a_2-q\right)d_{n+1} \leq\ \left(q-a_3-a_4\right)d_n$$

or $$d_{n+1} \leq (q-a_3-a_4)d_n \big/ \left(a_1+\ a_2-q\right)$$

or $$d_{n+1} \leq r\, d_n\ ,\ \ \text{where}\ \ r=\frac{q-a_3-a_4}{a_1+a_2-q}<1.$$

Therefore, $d_{n+1} \leq\ r\ d_n \leq\ r^2 d_{n-1} \leq\ \ldots\ldots\ \leq\ r^{n+1} d_0 \to\ 0\ as\, n\ \to \infty.$

Thus $\left\{y_n\right\}$ is a Cauchy sequence in X and by the completeness of X , $\left\{y_n\right\}$ converges to a point y (say) in X . Therefore, the sequences $\left\{fx_n\right\}$ and $\left\{gx_n\right\}$ also converge to y . Again since f and g commute weakly with each other, therefore

$$d\left(fgx_{n+1},\ gfx_{n+1}\right) \leq d\left(fx_{n+1},\ gx_{n+1}\right)=d\left(y_{n+1},\ y_{n+2}\right) \to 0\ \ \text{as}\ n \to \infty.$$

Now
$lim_{n\to\infty}\, gfx_{n+1}=lim_{n\to\infty} fgx_{n+1}=f\,(lim_{n\to\infty} gx_{n+1})=fy$ (since f is continuous).
Putting $x\ =\ fx_{n+1}$ in (3.1) and using the condition of weak commutativity, we have

$$a_1 d\left(g\left(fx_{n+1}\right),\, gy\right) + a_2\left(f\left(fx_{n+1}\right),\, g\left(fx_{n+1}\right)\right)$$
$$+a_3 d\left(gy,\, fy\right) + a_4 d\left(f\left(fx_{n+1}\right),\, fy\right)$$

$$\leq q \max\begin{Bmatrix} d\left(f\left(fx_{n+1}\right), fy\right),\, d\left(f\left(fx_{n+1}\right),\, gy\right), \\ d\left(g\left(fx_{n+1}\right), fy\right),\, d\left(g\left(fx_{n+1}\right),\, gy\right) \end{Bmatrix}.$$

In the limiting case, we get $\left(a_1 + a_3 - q\right) d\left(fy,\, gy\right) \leq 0$.

Therefore, $fy = gy$ since $a_1 + a_3 > q$, i.e., y is a coincidence point of f and g. Further we have by the condition of weak commutativity

$$d\left(f\left(g\left(y\right)\right),\, g\left(f\left(y\right)\right)\right) \leq d\left(fy,\, gy\right) = 0.$$

Therefore f(g(y)) = g(f(y)) = g(g(y)) i.e., g(y) is a coincidence point of f and g. Similarly $f(y)$ is a coincidence point of f and g. Now from (3.1); we have

$$a_1 d\left(g\left(g\left(y\right)\right),\, gy\right) + a_2 d(f\left(g\left(y\right)\right),\, g\left(g\left(y\right)\right)$$
$$+ a_3 d\left(fy,\, gy\right) + a_4 d\left(f\left(g\left(y\right)\right),\, fy\right) \leq 0.$$

i.e., $\left(a_1 + a_4 - q\right) d\left(g\left(g\left(y\right)\right),\, g\left(y\right)\right) \leq 0$

Therefore, $g\left(g\left(y\right)\right) = g\left(y\right)$ since $a_1 + a_4 > q$.

Thus g(y) is a fixed point of g and hence it is a common fixed point of both f and g.

Using (3.1) it can be easily shown that $g\left(y\right)$ is a unique common fixed point of f and g.

Taking $a_4 = 0$ and omitting $d\left(fx,\, fy\right)$ within max expression from the right hand side of theorem 3.1, we get the following result as a corollary of the above theorem.

Theorem 3.2. *Let $\left(X,\, d\right)$ be a complete metric space. Let f and g be a pair of weakly commuting self mappings of X with f continuous and*

$g(X) \subseteq f(X)$. *Let g satisfy*

$$a_1 d(gx, gy) + a_2 d(fx, gx) + a_3 d(fy, gy)$$
$$\leq q \max\{d(fx, gy), d(gx, fy), d(gx, gy)\},$$

for all $x, y \in X$ *with* $a_i \geq 0, q > 0$, $a_i > q+1, i = 1, 2, 3$ *and* $q < (a_1 + a_2 + a_3)/2$. *Then* f *and* g *have a unique common fixed point in* X .

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