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On P-Sasakian Manifolds Satisfying Certain Conditions on the Pseudo-Projective Curvature Tensor

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Abstract: In this paper we study the geometry of P-Sasakian manifolds under certain conditions like $R(X,\xi).\overline{P}=0$, $\overline{P}(\xi,X).S=0$ and $\overline{P}(X,Y)Z=0$. It is proved that in each case the manifold is either Einstein or $\eta-$Einstein.

Key words: Para-Sasakian manifolds, pseudo-projective curvature tensor, pseudo-projectively flat.

1. Introduction

In 1977 T. Adati and K. Matsumoto[1] introduced the notion of para-Sasakian manifolds or briefly P-Sasakian manifolds, which are considered as special cases of an almost para-contact manifold introduced by I. Sato[2]. The notion of semi-symmetric manifold is defined by $R(X,Y).R=0$. It is studied by many authors some of them are De and Kamilya[3], Perrone[4] and Szabo[5]. De and Pathak[7] also have studied $\mathrm{R}(\mathrm{X},\mathrm{Y}).P=0$ and $R(X,Y).S=0$ in P-Sasakian manifolds. In this paper we study some derivation conditions on P-Sasakian manifolds. In preliminaries we give a brief account of P-Sasakian manifolds, pseudo-projective curvature tensor $\overline{P}$ and some basic results on P-Sasakian manifolds. In section 3, we study the condition $R(X,\xi).\overline{P}=0$ and it is proved that the manifold is $\eta-$Einstein. Section 4 deals with the condition $\overline{P}(\xi,X).S=0$ and it is proved that the manifold is Einstein. In last section, we study the pseudo-projectively flat P-Sasakian manifold and it is shown that such a manifold is Einstein.

2. Preliminaries

An n-dimensional smooth manifold M is called an almost paracontact Riemannian manifold if it admits an almost paracontact Riemannian structure (ϕ,ξ,η,g) consisting of (1, 1) tensor field ϕ, a contravarient vector field ξ, a 1-form η and an associated Riemannian metric g satisfying

(2.1) $$\phi^2(X)=X-\eta(X)\xi,$$

(2.2) $$\eta(\xi)=1,$$

(2.3) $$\phi\xi=0,\quad \eta\circ\xi=0,\quad \text{rank}\quad \phi=n-1,$$

(2.4) $$\eta(X)=g(\xi,X),\quad \eta(\phi X)=0,$$

(2.5) $$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),$$

(2.6) $$g(X,\phi Y)=g(\phi X,Y).$$

In addition, if the structure (ϕ,ξ,η,g) satisfies the equations

(2.7) $$(\nabla_X\eta)(Y)=(\nabla_Y\eta)(X)=g(X,\phi Y),$$

(2.8) $$\nabla_X\xi=\phi X,\quad d\eta=0,$$

Then M is called a para-Ssasakian manifold or briefly a P-Sasakian manifold[1]. In a P-Sasakian manifold the following fundamental relations hold[1,2,6,7,8]:

(2.9) $$R(X,Y)\xi=\eta(X)Y-\eta(Y)X,$$

(2.10) $$R(\xi,X)Y=\eta(Y)X-g(X,Y)\xi=-R(X,\xi)Y,$$

(2.11) $$S(X,\xi)=-(n-1)\eta(X),$$

(2.12) $$S(\xi,\xi)=-(n-1),$$

(2.13) $$\eta(R(X,Y)Z)=g(R(X,Y)Z,\xi)=g(X,Z)\eta(Y)-g(Y,Z)\eta(X),$$

(2.14) $$\eta(R(X,Y)\xi)=0$$

for any vector fields $X, Y, Z,$ where $R(X,Y)Z$ is the Riemannian curvature tensor.

A P- Sasakian manifold is said to be an Einstein and an $\eta-$Einstein manifold if its Ricci tensor S is of the form

(2.15) $S(X,Y) = \alpha g(X,Y)$ and $S(X,Y) = \beta g(X,Y) + \gamma\eta(X)\eta(Y),$

respectively, where α is a constant and β and γ are smooth functions on M. The pseudo-projective curvature tensor $\bar{P}$ on a manifold M of dimension n is defined by Bhagawat[9]

(2.16)
$$\bar{P}(X,Y)Z = aR(X,Y)Z + b\left[S(Y,Z)X - S(X,Z)Y\right] - \frac{r}{n}\left\{\frac{a}{n-1} + b\right\}\left[g(Y,Z)X - g(X,Z)Y\right]$$

where a, b are constants such that $a, b \neq 0$ and R, S, r are the curvature tensor, Ricci tensor and scalar curvature respectively.

3. P-Sasakian manifolds satisfying $\mathbf{R(X,\xi).\bar{P} = 0}$

Let us consider a P-Sasakian manifold (M^n, g) satisfying the condition

(3.1) $$R(X,\xi).\bar{P} = 0.$$

From (2.13) & (2.16) we have

(3.2)
$$\eta(\bar{P}(U,V)W) = \left[a + \frac{\{a + (n-1)b\}r}{n(n-1)}\right]\left[g(U,W)\eta(V) - g(V,W)\eta(U)\right] + b\left[S(V,W)\eta(U) - S(U,W)\eta(V)\right].$$

Putting $W = \xi$ in (3.2) and using (2.4) and (2.11) we get

(3.3) $$\eta(\bar{P}(U,V)\xi) = 0.$$

Again taking $U = \xi$ in (3.2) and using (2.2) and (2.4) we obtain

$$\eta\left(\bar{P}(\xi,V)W\right)=\left[a+\frac{\{a+(n-1)b\}r}{n(n-1)}\right]\left[\eta(W)\eta(V)-g(V,W)\right] \tag{3.4}$$
$$+b\left[S(V,W)+(n-1)\eta(V)\eta(W)\right].$$

Now

$$\left(R(X,\xi).\bar{P}\right)(U,V)W=R(X,\xi)\bar{P}(U,V)W-\bar{P}\left(R(X,\xi)U,V\right)W \tag{3.5}$$
$$-\bar{P}\left(U,R(X,\xi)V\right)W-\bar{P}(U,V)R(X,\xi)W.$$

From (3.1) and (3.5), we have

$$R(X,\xi)\bar{P}(U,V)W-\bar{P}\left(R(X,\xi)U,V\right)W-\bar{P}\left(U,R(X,\xi)V\right)W \tag{3.6}$$
$$-\bar{P}(U,V)R(X,\xi)W=0$$

By virtue of (2.10) and (3.6), we get

$$g\left(\bar{P}(U,V)W,X\right)-\eta(X)\eta\left(\bar{P}(U,V)W\right)-g(X,U)\eta\left(\bar{P}(\xi,V)W\right)$$
$$+\eta(U)\eta\left(\bar{P}(X,V)W\right)-g(X,V)\eta\left(\bar{P}(U,\xi)W\right)+\eta(V)\eta\left(\bar{P}(U,X)W\right) \tag{3.7}$$
$$+\eta(W)\eta\left(\bar{P}(U,V)X\right)=0,$$

since by (3.3) $\qquad \eta\left(\bar{P}(U,V)\xi\right)=0.$

Putting X=U in (3.7), we get

$$g\left(\bar{P}(U,V)W,U\right)-g(U,V)\eta\left(\bar{P}(\xi,V)W\right)-g(U,V)\eta\left(\bar{P}(U,\xi)W\right) \tag{3.8}$$
$$+\eta(V)\eta\left(\bar{P}(U,U)W\right)+\eta(W)\eta\left(\bar{P}(U,V)U\right)=0.$$

Let $\{e_i : i=1,2,....,n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $U=e_i$ in (3.8) and taking summation for $1\le i\le n$ we get

$$\sum_{i=1}^{n} g\left(\bar{P}(e_i,V)W,e_i\right)-(n-1)\eta\left(\bar{P}(\xi,V)W\right)+\eta(W)\sum_{i=1}^{n}\eta\left(\bar{P}(e_i,V)e_i\right)=0. \tag{3.9}$$

From (2.16) and (3.2), it follows that

(3.10) $$\sum_{i=1}^{n} g\left(\bar{P}(e_i,V)W,e_i\right)=\left[a+(n-1)b\right]S(V,W)-\left[\left\{\frac{a+(n-1)b}{n}\right\}r\right]g(V,W),$$

(3.11) $$\sum_{i=1}^{n} \eta(W)\eta\left(\bar{P}(e_i,V)e_i\right)=\left[(a-b)\left\{(n-1)+\frac{r}{n}\right\}\right]\eta(V)\eta(W).$$

Using (3.10) and (3.11) in (3.9) we obtain

(3.12) $$\eta\left(\bar{P}(\xi,V)W\right)=\left[\frac{a+(n-1)b}{n-1}\right]S(V,W)-\left[\frac{\{a+(n-1)b\}r}{n(n-1)}\right]g(V,W) + \left[\frac{a-b}{n-1}\left\{(n-1)+\frac{r}{n}\right\}\right]\eta(V)\eta(W).$$

From (3.4) and (3.12), we get

(3.13) $$S(V,W)=-(n-1)g(V,W)+\left[\frac{b}{a}\{r+n(n-1)\}\right]\eta(V)\eta(W).$$

The relation (3.13) implies that the manifold is an $\eta-$Einstein.
Hence we can state the following:

Theorem 3.1: *A P-Sasakian manifold* (M^n,g) *satisfying the condition* $R(X,\xi).\bar{P}=0$ *is an* $\eta-$*Einstein manifold provided* $a-b\neq 0$.

Taking an orthonormal frame field and contracting (3.13) over V and W we obtain

(3.14) $$r=-n(n-1) \text{ if } a-b\neq 0.$$

Using (3.14) in (3.13), we get

(3.15) $$S(V,W)=-n(n-1)g(V,W).$$

This leads to the following result:

Theorem 3.2*: A P-Sasakian manifold* (M^n, g) *satisfying the condition* $R(X,\xi).\bar{P}=0$ *is an Einstein manifold and is also a manifold of constant negative scalar curvature* $-n(n-1)$.

4. P-Sasakian manifold satisfying $\bar{\mathrm{P}}(\xi,\mathrm{X}).\mathrm{S}=0$

Let M be an n-dimensional P-Sasakian manifold, which satisfies

$$\bar{\mathrm{P}}(\xi,\mathrm{X}).\mathrm{S}=0,$$

then we have

$$(\bar{P}(\xi,X).S)(Y,\xi)=0. \tag{4.1}$$

This implies that

$$S(\bar{P}(\xi,X)Y,\xi)+S(Y,\bar{P}(\xi,X)\xi)=0. \tag{4.2}$$

In view of (2.4), (2.10), (2.11), (2.12) and (2.16) in (4.2) we obtain

$$\begin{aligned}&\left[a+\frac{\{a+(n-1)b\}r}{n(n-1)}\right]\left[\eta(Y)S(X,\xi)-g(X,Y)S(\xi,\xi)\right]\\ &+bS(X,Y)S(\xi,\xi)+(n-1)b\eta(Y)S(X,\xi)\\ &+\left[\{a+(n-1)b\}\left\{1+\frac{r}{n(n-1)}\right\}\right]\left[S(X,Y)-\eta(X)S(Y,\xi)\right]=0.\end{aligned} \tag{4.3}$$

Using (2.11) and (2.12) in last relation we get

$$\begin{aligned}&\left[a+\frac{\{a+(n-1)b\}r}{n(n-1)}\right]\left[-(n-1)\eta(X)\eta(Y)+(n-1)g(X,Y)\right]\\ &-(n-1)bS(X,Y)-(n-1)^2b\eta(X)\eta(Y)\\ &+\left[\{a+(n-1)b\}\left(1+\frac{r}{n(n-1)}\right)\right]\left[S(X,Y)+(n-1)\eta(X)\eta(Y)\right]=0.\end{aligned} \tag{4.4}$$

This relation on further simplification yields

$$S(X,Y) = -(n-1)\,g(X,Y). \tag{4.5}$$

Hence the manifold is Einstein.
Now, we can state:

Theorem 4.1*: A P-Sasakian manifold* (M^n, g) *satisfying the condition* $\overline{P}(\xi, X).S = 0$ *is an Einstein manifold.*

5. Pseudo-Projectively flat P-Sasakian manifold

Let us consider a P-Sasakian manifold (M^n, g) which is pseudo-projectively flat. Then we have $\overline{P}(X,Y)Z = 0.$ Now from (2.16) we have

$$\begin{aligned} a\overline{R}(X,Y,Z,W) = &-b\left[S(Y,Z)\,g(X,W) - S(X,Z)\,g(Y,W)\right] \\ &+\left[\frac{\{a+(n-1)b\}r}{n(n-1)}\right]\left[g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W)\right], \end{aligned} \tag{5.1}$$

where $\overline{R}(X,Y,Z,W) = g\left(R(X,Y)Z, W\right).$
Setting $W = \xi$ is (5.1) we get

$$\begin{aligned} \eta\left(R(X,Y)Z\right) = &-\frac{b}{a}\left[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)\right] \\ &+\left[\frac{\{a+(n-1)b\}r}{n(n-1)a}\right]\left[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\right]. \end{aligned} \tag{5.2}$$

By virtue of (2.13) and (5.2) we obtain

$$\begin{aligned} -S(Y,Z)\eta(X) + S(X,Z)\eta(Y) = &\frac{a}{b}\left[1 + \frac{\{a+(n-1)b\}r}{n(n-1)a}\right] \\ &\left[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\right]. \end{aligned} \tag{5.3}$$

Replacing Y by ξ in (5.3) and using (2.11) and (2.2) we have

$$(n-1)\eta(X)\eta(Z)+S(X,Z)=\left[\frac{a}{b}+\frac{\{a+(n-1)b\}r}{n(n-1)b}\right]g(X,Z)$$
$$-\left[\frac{a}{b}+\frac{\{a+(n-1)b\}r}{n(n-1)b}\right]\eta(X)\eta(Z).$$

From last relation we obtain

(5.4)
$$S(X,Z)=\left[\frac{a}{b}+\frac{\{a+(n-1)b\}r}{n(n-1)b}\right]g(X,Z)-$$
$$\left[\left\{\frac{a+(n-1)b}{b}\right\}\left\{1+\frac{r}{n(n-1)}\right\}\right]\eta(X)\eta(Z).$$

Let $\{e_i : i=1,2,.....,n\}$ be an orthonormal basis of tangent space at any point of the manifold. Setting $X=Z=e_i$ in (5.4) and taking summation for $1\le i\le n,$ we get

(5.5) $$r=-n(n-1).$$

Using (5.5) in (5.4), we obtain

(5.6) $$S(X,Z)=-(n-1)\,g(X,Z).$$

This leads to the following result:

Theorem 5.1: *A pseudo-projectively flat P-Sasakian manifold* (M^n, g) *is an Einstein manifold .*

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