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Some Characterization of H_μ Type Spaces

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Abstract: In this paper we prove that $H_{\mu,M} = H^p_{\mu,M}$, $H_{\mu,M,a} = H^p_{\mu,M,a}$ and $H^\Omega_\mu = H^{\Omega,p}_\mu$, $H^{\Omega,b}_\mu = H^{\Omega,b,p}_\mu$ for $1 \le p < \infty$ by using the theory of Hankel transformation.

Keyword: L^p-space, Hankel transform, convex function.
AMS classification: 46F12.

1. Introduction

The classical Hankel transform of a function $\phi \in L^1(0,\infty)$ is defined by $\left(h_\mu \phi\right)(y) = \int_0^\infty \phi(x)\,(xy)^{\frac{1}{2}}\, J_\mu(xy)dx,\ \mu \ge -\frac{1}{2}$ and was extended to distribution by Ziemanian[1]. Motivated from the results given in Gel'f and and Shilov[2], Pathak[3] and Pathak-Sahoo[4] introduced the spaces of type U_μ and H_μ and studied their Hankel transform and extended the results to distributions by adjoint method. Their $L^p(0,\infty)-$type spaces was studied by Pathak and Upadhyay[5].

Now, we recall the definitions of the spaces of type $H^p_{\mu,M,a}$, $H^{\Omega,b}_\mu, H^p_{\mu,M,a}, H^{\Omega,b,p}_\mu$ from Pathak and Sahoo[4], Pathak and Upadhyay[5].

Let M and Ω be the convex functions which are defined by

$$M(x) = \int_0^x \mu(\xi)d\xi, x \ge 0 \text{ and } \Omega(y) = \int_0^y \omega(\eta)d\eta, y \ge 0, \tag{1}$$

$$M(x) = M(-x), M(x_1) + M(x_2) \le M\left(x_1 + x_2\right), \tag{2}$$

and

(3) $$\Omega(y)=\Omega(-y), \Omega(y_1)+\Omega(y_2)\le\Omega\left(y_1+y_2\right).$$

Now, the space $H_{\mu,M}$, is the set of all infinitely differentiable functions ϕ on $I=(0,\infty)$ which satisfy the inequalities

(4) $$\left|S^k_{\mu,x}\phi(x)\right|\le C_k \exp\left[M[ax]\right].$$

The space $H^{\Omega,b}_{\mu}$ consists of all even entire functions ϕ such that for every b>0, $k\in \mathbf{N_0}$ there exists $C_k>0$ such that

(5) $$\left|z^{2k-\mu-\frac{1}{2}}\phi(z)\right|\le C_k \exp\left[\Omega[by]\right].$$

The function ϕ is in $H_{\mu,M,a,}$ if and only if, for each a>0, δ>0 there exists $C_{k,\delta}>0$ such that

(6) $$\left|S^k_{\mu,x}\phi(x)\right|\le C_{k,\delta} \exp[-M[(a-\delta)x].$$

Even entire analytic function $z^{2k-\mu-\frac{1}{2}}\phi(z)$ is in $H^{\Omega,b,p}_{\mu}$ if and only if b > 0, $\rho>0$ and $C_{k\rho,}>0$ such that

(7) $$|z^{2k-\mu-\frac{1}{2}}\phi(z)|\le C_{k,\rho}\exp[\Omega[(b+\rho)y]] .$$

Like Pathak and Upadhyay[5] the spaces of type $H^p_{\mu,M}, H^p_{\mu,M,a}$, and $H^{\Omega,b,p}_{\mu}$ are defined as follows:

(i) A complex valued and smooth functions $\phi=\phi(x),\ x\in I=(0,\infty)$ is in $H^p_{\mu,M}$ if and only if for $C_{k,p}>0$, $a>0$ such that

(8) $$\left(\int_0^\infty|\exp[M(ax)]S^K_{\mu,x}\phi(x)|^p\,dx\right)^{1/p}\le C_{k,p}.$$

A infinitely differentiable smooth function ϕ is in $H^p_{\mu,M,a}$ if it satisfies the inequalities

(9) $$\left(\int_0^\infty|\exp[M[a-\delta)x]]\ S^k_{\mu,x}\phi(x)|^p\ dx\right)^{\frac{1}{p}}\le C_{k,\delta,p}.$$

for $k>0$, $\delta>0$.

Even entire analytic function $z^{2k-\mu-\frac{1}{2}}\phi(z)$ belongs to $H_\mu^{\Omega,p}$ if and only if for $k>0$, $a>0$ and $C_{k,p}>0$ such that

$$(10)\qquad \left(\int_0^\infty |\exp([\Omega(by)]z^{2k-\mu-\frac{1}{2}}\phi(z)|^p\, dx\right)^{\frac{1}{p}} \le C_{k,p}.$$

(iii) $z^{2k-\mu-\frac{1}{2}}\,\phi(z)\in H_\mu^{\Omega,b,p}$ iff for $b>0$, $\rho>0$, $C_{k,\rho,p}>0$

such that

$$(11)\qquad \left(\int_0^\infty |\exp(\Omega[(b+\rho)y]z^{2k-\mu-\frac{1}{2}}\phi(z)|^p\, dx\right)^{\frac{1}{p}} \le C_{k,\rho,p}.$$

Using the aforesaid results we establish that

$$H_{\mu,M} = H_{\mu,M}^p,\; H_{\mu,M,a} = H_{\mu,M,a}^p\ ,$$

and

$$H_\mu^\Omega = H_\mu^{\Omega,p},\; H_\mu^{\Omega,b} = H_\mu^{\Omega,b,p} \text{ for } \mu \ge -\frac{1}{2} \text{ and } 1\le p<\infty.$$

2. Characterization of H_μ- type spaces

In this section we study the relation between $H_{\mu,M}$, $H_{\mu,M,a}$, H_μ^Ω, $H_\mu^{\Omega,b}$ and $H_{\mu,M}^p$, $H_\mu^{\Omega,p}$, $H_\mu^{\Omega,p}$ for $1\le p<\infty$ by using the theory of Hankel transformation.

To find this relation, The Young inequality is useful which can be defined by the following way:

$$(12)\qquad xy \le M(x)+\Omega(y),$$

where $M(x)$ and $\Omega(y)$ are pair of dual in sense of Young.

Theorem 2.1. *Let M(x), Ω(y) be the pair of functions which are dual in sense of Young. Then*

$$H_{\mu,M,a} = H_{\mu,M,a}^p \text{ for } 1\le p<\infty.$$

Proof. Let $\phi \in H^{p}_{\mu,M,a}$. Then Hankel transformation exists in $L^{1}(0,\infty)$ sense and $s^{-\mu-\frac{1}{2}}\,\psi(s)$ is an even entire analytic function[4], we can write

$$(-1)^{k} s^{2k}\psi(s) = \int_{0}^{\infty} S^{k}_{\mu,x}\phi(x)\ (xs)^{\frac{1}{2}} J_{\mu}(xs)dx\ ,$$

for s = u+it. Therefore, from [3, p. 138], we have

$$\begin{aligned}\left|s^{2k-\mu-\frac{1}{2}}\psi(s)\right| \ \le\ & \int_{0}^{\infty}\left|e^{-x|t|}(xs)^{-\mu}J_{\mu}(xs)\right| \\ & \left|\exp[M[(a-\delta)x]]S^{k}_{\mu,x}\phi(x)\right| \\ & \left|x^{\mu+\frac{1}{2}}\exp[x\,|\,t\,|-M[(a-\delta)x]]\right|dx \\ \le\ & A_{\mu}\left\|\exp[M[(a-\delta)x]\ S^{k}_{\mu,x}\phi(x)\right\|_{p} \\ & \left\|x^{\mu+\frac{1}{2}}\ \exp[x\,|\,t\,|-M[(a-\delta)x]]\right\|_{q}\ ,\end{aligned}$$

for

$$\frac{1}{p}+\frac{1}{q}=1 \text{ and } 1\le p,q<\infty.$$

Now, using the technique of Gel'fand and Shilov[2], Pathak and Sahoo[4] we have

$$\begin{aligned}\left|s^{2k-\mu-\frac{1}{2}}\ \psi(s)\right| \ \le\ & D_{k,\mu,\rho,p}\ \exp\left[\Omega\left(\frac{1}{a}+\rho\right)\right] \\ & \left\|x^{\mu+\frac{1}{2}}\exp\left[M\left(\delta x\right)\right]\right\|_{q}.\end{aligned}$$

Therefore

$$\psi(s)\in H^{\Omega,\frac{1}{a}}_{\mu}.$$

This implies that

$$h_{\mu}\left[H^{p}_{\mu,M,a}\right]\subset H^{\mu,1/a}.$$

By property of inverse Hankel transform, we have

$$H^{p}_{\mu,M,a} \subseteq h_\mu^{-1}\left[H_\mu^{\Omega,\frac{1}{a}}\right] \subseteq H_{\mu,M,a}.$$

Hence

(13) $$H^{p}_{\mu,M,a} \subseteq H_{\mu,M,a}.$$

From Pathak and Sahoo's[4] Theorem 4.1

$$h_\mu[H_{\mu,M,a}] \subset H_\mu^{\Omega,\frac{1}{a}}.$$

From the definition of inverse Hankel transform, we have

(14) $$H_{\mu,M,a} \subseteq h_\mu^{-1}\left[H_\mu^{\Omega,\frac{1}{a}}\right].$$

Now, we take $\phi \in H_\mu^{\Omega,\frac{1}{a}}$ then from Pathak and Sahoo's[4]

$$S^{k}_{\mu,u}\phi(u) = \int_0^\infty x^{2q}\ (-1)^q\ \psi(x)\ (xu)dx.$$

By Young inequality (12) and the technique[4], we find that

$$\left|S^{k}_{\mu,u}\ \phi(u)\right| \le C_{q,\delta}\ \exp\left[-M\left[(a\text{-}\delta)u\right] - M\left(a^3\rho^2\right)u\right].$$

Therefore,

$$\left\|\exp\left[M\left[(a-\delta)u\right]\right]\ S^{k}_{\mu,u}\ \phi(u)\right\|_p \le C_{q,\delta,p}\left\|\exp\left[-M\left(a^3\rho^2\right)u\right]\right\|_p.$$

This implies that

(15) $$h_\mu^{-1}\left[H^{\Omega,\frac{1}{a}}\right] \subseteq H^{p}_{\mu,M,a}.$$

From (14) and (15), we have

(16) $$H_{\mu,M,a} \subseteq H^{p}_{\mu,M,a}.$$

So that, (13) and (16) gives

$$H_{\mu,M,a} = H^{p}_{\mu,M,a}.$$

Theorem 2.2. *Let M and Ω be the functions which are dual in sense of Young. Then for* $\mu \geq -\frac{1}{2}$

$$H_{\mu}^{\Omega,b} = H_{\mu}^{\Omega,b,p}.$$

Proof. Let $\phi \in H_{\mu}^{\Omega,b,p}$. Then we can write[4]

$$S_{\mu,u}^{k}\psi(u) = \int_{0}^{\infty} \phi(x)\ (-1)^{q}\ x^{2q}(1+x^{2})\ \left(1+x^{2}\right)^{-1}\ (xu)^{\frac{1}{2}} J_{\mu}(xu) d\sigma(u).$$

Therefore

$$\begin{aligned}
\left|S_{\mu,u}^{k}\psi(u)\right| &\leq A_{\mu} \left\|\left(1+x^{2}\right)x^{2q}\ \phi(x)\right\|_{p}\ \left\|\left(1+x^{2}\right)^{-1}\right\|_{q} \\
&\leq A_{\mu}\ A_{q}\ \left\|\left(1+x^{2}\right)x^{2q}\phi(x)\right\|_{p} \\
&= A_{q,p}\left\|\left(1+x^{2}\right)\ x^{2q+\mu+\frac{1}{2}}\ x^{-\mu-\frac{1}{2}}\ \phi(x)\right\|_{p} \\
&\leq A_{q,p}\ \left\|\exp[-\sigma|y|]\ \left(1+z^{2}\right)^{r+1} z^{-\mu-\frac{1}{2}}\phi(z)\right\|_{p}
\end{aligned}$$

for $r > 2q + \mu + \frac{1}{2}$.

This implies that

$$\begin{aligned}
\left|S_{\mu,u}^{k}\phi(u)\right| &\leq A_{q,p} \exp[-\sigma\,|\,y\,|] \sum_{n=0}^{\gamma+1} \binom{\gamma+1}{n} \|\, z^{2n-\mu-\frac{1}{2}}\, \phi(z)\,\|_{p} \\
&\leq A_{q,p} \exp[-\sigma\,|\,y\,| + \Omega[(b+\rho)y]] \sum_{n=0}^{\gamma+1} C_{n} \\
&\leq C_{q,\gamma+1,p} \exp[-\sigma\,|\,y\,| + \Omega[(b+\rho)y].
\end{aligned}$$

From Pathak and Sahoo[4], we have

$$|\, S_{\mu,u}^{k}\ \phi(u)\,| \leq C_{q,r+1,p} \exp\left[-M\left[\left(\frac{1}{b}-\delta\right)u\right]\right].$$

Hence

$$h_\mu\left[H^{\Omega,b,p}\right]\subseteq H_{\mu,M,\frac{1}{b}}.$$

Therefore

$$H^{\Omega,b,p}\subseteq h_\mu^{-1}\left[H_{\mu,M,\frac{1}{b}}\right]\subseteq H^{\Omega,b}.$$

This implies that

(17) $$H^{\Omega,b,p}\subseteq H^{\Omega,b},$$

Now, we have to prove that

$$H_\mu^{\Omega,b}\subseteq H^{\Omega,b,p}.$$

We take $\phi\in H_{\mu,M,\frac{1}{b}}$ and using the arguments[4]

$$\left|s^{2k-\mu-\frac{1}{2}}\psi(s)\right|\leq C'''_{k,\delta}\exp\left[\left[\Omega\left[\left(\frac{1}{b}+\rho\right)t\right]\right]-M\left(\frac{\rho u}{b^3}\right)\right].$$

Therefore

$$\left\|s^{2k-\mu-\frac{1}{2}}\psi(s)\right\|_r\leq C'''_{k,\delta,r}\exp\left[\Omega\left[\left(\frac{1}{b}+\rho\right)t\right]\right],$$

$$\left\|\exp\left[-M\left(\frac{\rho u}{b^3}\right)\right]\right\|_r.$$

Then

$$h_\mu\left[H_{\mu,M,\frac{1}{b}}\right]\subseteq H_\mu^{\Omega,b,p}.$$

Since, from Pathak and Sahoo's[4] Theorem 4.1, we have

$$h_\mu\left[H_{\mu,M,\frac{1}{b}}\right]\subseteq H_\mu^{\Omega,b}.$$

Hence

(18) $$H_{\mu}^{\Omega,b} \subseteq H_{\mu}^{\Omega,b,p}.$$

Thus, from (17) and (18), we can prove that

$$H_{\mu}^{\Omega,b} = H^{\Omega,b,p}.$$

Theorem 2.3. $H_{\mu,M} = H_{\mu,M}^{p}$, $H_{\mu}^{\Omega,p} = H_{\mu}^{\Omega}$, $\mu \geq -\frac{1}{2}$.

Proof. From Theorem 2.1, we have

$$H_{\mu,M,a} = H_{\mu,M,a}^{p} \quad \forall\ a > 0.$$

Since the spaces $H_{\mu,M}, H_{\mu}^{\Omega,b}$, $H_{\mu}^{\Omega,p,p}$ can be regarded as union of normed linear spaces $H_{\mu,M,a}$, $H_{\mu,M,a}^{p}$, $H_{\mu}^{\Omega,b}$, $H_{\mu}^{\Omega,b,p}$. Therefore, from Theorem 2.1 and Theorem 2.2, we can write

$$\bigcup_{a} H_{\mu,M,a} = \bigcup_{a} H_{\mu,M,a}^{p}.$$

and

$$\bigcup_{a} H_{\mu}^{\Omega,b} = \bigcup_{a} H_{\mu}^{\Omega,b,p}.$$

This implies that

$$H_{\mu,M} = H_{\mu,M}^{p} \ \text{ and } \ H_{\mu}^{\Omega} = H_{\mu}^{\Omega,p}.$$

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