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# Minimal Hypersurface of a Finsler Space with Recurrent Connection

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**Abstract:** This paper characterizes minimal hypersurface with respect to recurrent generalized cartan connection.
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## 1. Introduction

The theory of hypersurface in a Finsler space has been considered first by E. Cartan[1] from two points of view. One is to regard a hypersurface as the whole of tangent line- elements and then it is also a Finsler space[2]. The other is to regard it as the whole of normal line- elements and then it is a Riemannian space. J. M. Wegener[3-4] has treated 'hypersurfaces from the latter view point and dealt in particular with minimal hypersurface. E.T. Davies[5] has considered subspaces from the former view point mainly, but referred a little to minimal subspaces. Both of them have pointed out a weak point of their theories that the minimal subspaces are characterized only by the vanishing of the mean curvalture provided Cartan's torsion vector vanishes.

M. Matsumoto[6] has proposed a new Finsler connection with surviving torsion tensor, called Cartan's Y- connection and obtained the condition for the hypersurface to be a minimal hypersurface.

In almost all the above theories it has been supposed that the connection is metrical, so that the convariant differentiation commutes with raising and lowering of indices. In 1990 B. N. Prasad et al[7] have introduced a Finsler connection with respect to which thi metric tensor is h- recurrent. Such a Finsler connection has been called h- recurrent Finsler connection. While introducing h-recurrent Finsler connection they have assumed that the v-covariant derivative of the metric tensor vanishes and torsion fields T and S vanish.

In the paper[8] we have obtained the Y-extremal hypersurface of a Finsler space whose connection is h-recurrent with non-vanishing torsion. The purpose of the present paper is to obtain the condition for hypersurface of a Finsler space with h-recurrent generalized Cartan connection to be minimal hypersurface.

## 2. Preliminaries

Let $F^n = (M^n, L, F\Gamma)$ be a Finsler space on an n-dimensional underlying manifold $M^n$, equipped with a fundamental function $L = L(x, y)$ and a Finsier connection[9, 10] $F\Gamma = (\Gamma, N, C)$.

A Finisler connection $F\Gamma = (F^{\ i}_{jk}, N^{\ i}_{j}, C^{\ i}_{jk})$ is called a recurrent generalized Cartan connection and is denoted by Rec $C\Gamma(a, T)$ if the following five axioms are satisfied:

(F-1) $F\Gamma$ is h-recurrent for a given covariant vector field $a_{k,}$ i.e. $g_{ij/k} = a_k\, g_{ij}$,

(F-2) $F\Gamma$ is v-metrical,i.e $g_{ij}\,|_k = 0$,

(F-3) Deflection tensor field vanishes, i.e. $N^i_k = F^i_{jk}\, y^j$,

(F-4) The (v) v- torsion tensor field vanishes, i.e. $C^i_{jk} = C^i_{kj}$,

(F-5) The (h) h-torsion tensor $T^{\ i}_{jk} (= F^{\ i}_{jk} - F^{\ i}_{kj})$ is a given tensor field.

The fundamental metric tensor is defined by

$$g_{ij}(x, y) = 1/2\, \dot{\partial}_i \dot{\partial}_j L^2(x, y),$$

and $a_k$ in (F-1) are components of a covariant vector field depending upon x and y. Conditions (F-2) and (F-4) lead to $C_{ijk} = C^{\ r}_{ik}\, g_{rj} = \frac{1}{2}\dot{\partial}_k\, g_{ij}$,

whereas condition (F-1) leads to

$$
\begin{aligned}
(2.1) \qquad F_{jik} &= F^{\ r}_{jk}\, g_{ri} \\
&= \gamma_{jik} - (C_{kim} N^{\ m}_{j} + C_{jim} N^{\ m}_{k} - C^{\ m}_{jk} N^{\ r}_{i}\, g_{rm}) + A_{jik} + B_{jik}
\end{aligned}
$$

where

(2.2) $$2\gamma_{jik} = \partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk},$$

(2.3) $$2A_{jik} = T_{jik} + T_{jki} + T_{kji}, \quad T_{jik} = T^{r}_{jk} g_{ri},$$

(2.4) $$2B_{jik} = a_i g_{jk} - a_j g_{ik} - a_k g_{ij.}$$

The condition (F-3) and (2.1) give $N^{i}_{j}$

## 3. The Unit Normal Vector

Let a hypersurface $M^{n-1} : x^i = x^i(u^\alpha), \alpha = 1, 2 \ldots\ldots\ldots, n-1,$ be given in the underlying manifold $M^n$ of an n-dimensional Finsler space $F^n = \left(M^n, L, \mathrm{Re}c\ C\Gamma(a, T)\right)$ equipped with a Finsier metric $L$ and a generalized recurrent Cartan connection $\mathrm{Re}c\ C\Gamma(a, T)$. The *(n-1)* tangent vectors $B_\alpha = \left(B^i_\alpha = \dfrac{\partial x^i}{\partial u^\alpha}\right)$ are assumed to be linearly independent. The combination $(x, B_\alpha)$ of a point $x(u)$ of $M^{n-1}$ and $(n-1)$ tangent vectors $B_\alpha(u)$ at $x(u)$ is called a hypersurface element of $F^n$.

The unit normal vector $N = (N^i)$ of a hypersurface element $(x, B_\alpha)$ is defined by the equations

(3.1) $$L(x, N) = 1 \text{ or } g_{ij}(x, N) N^i N^j = 1.$$

(3.2) $$g_{ij}(x, N) B^i_\alpha N^j = 0, \qquad \alpha = 1, 2, \ldots\ldots\ldots\ldots\ldots, n-1.$$

To construct the vector $N$ we take $n$ constants $d^i$ such that the square matrix $(B^i_\alpha, d^i)$ has non- zero determinant D. Let $q_i$ be the cofactor of $d^i$ in $(B^i_\alpha, d^i)$. Then $q_i = q_i(B)$ are functions of $B^i_\alpha$, independent of choice of $d^i$, and satisfy

(3.3) $$q_i B^i_\alpha = 0. \qquad q_i d^i = D.$$

Next then equations

(3.4) $$g_{ij}(x, p) p^j = q_i,$$

give $p^i$ uniquely. Thus we get n functions $p^j = p^j(x, q(B))$. We put

(3.5) $$N^i = \frac{p^i}{L(x,p)}.$$

Then $N^i$ are components of a contravariant vector and satisfy equations (3.1) and (3.2). Also from (3.5) it follows that $N^i$ are functions of x and B. The induced Riemannian N- metric on $M^{n-1}$ is given by

(3.6) $$g_{\alpha\beta}(u) = g_{ij}(x,N)\, B^i_\alpha\, B^i_\beta .$$

Then we have

(3.7) $$B^\alpha_i = g^{\alpha\beta}\, g_{ij}(x,N)\, B^j_\beta \quad \text{and} \quad N_i = g_{ij}(x,N)\, N^j.$$

From (3.4), (3.5) and (3.7) we have

(3.8) $$q_i = L(x,p)\, N_i .$$

We quote the following which has been derived in[6]

(3.9) $$\frac{\partial L(x, p(x,B))}{\partial B^i_\alpha} = L(x,p)\, B^\alpha_i ,$$

(3.10) $$\frac{\partial N^i(x,B)}{\partial B^i_\alpha} = -B^{\alpha i}\, N_{j,}$$

where

$$B^{\alpha i} = g^{ij}\, B^\alpha_j = g^{\alpha\beta}\, B^i_\beta ,$$

## 4. The Induced Connection

The absolute differential $DX$ of a tangent vector field $X^i = X^\alpha B^i_\alpha$ of $M^{n-1}$ is defined by $DX^\alpha = B^\alpha_i X^i$, where $DX^i$is the absolute differential of $X^i$ with respect to $\mathrm{Rec}\,C\Gamma(a,T)$ in which the supporting element $y^i$ is specified as the normal vector $N^i$. The connection coefficient $\Gamma^\alpha_{\beta\gamma}(u)$ of induced connection $\underline{\Gamma}$ are given by[6]

(4.1) $$\Gamma^{\alpha}_{\beta\gamma} = B^{\alpha}_{i}(B^{i}_{\beta\gamma} + B^{j}_{\beta} F^{i}_{j\gamma}),$$

where

(4.2) $$F^{i}_{j\gamma}(u) = F^{i}_{jk}(x, N) B^{k}_{\gamma} + C^{i}_{jk}(x, N)\ N^{k}_{\gamma},$$

(4.3) $$N^{k}_{\gamma} = \frac{\partial N^{k}}{\partial u^{\gamma}} N^{k}_{h}(x, N) B^{h}_{\gamma}.$$

Therefore we obtain the Gauss formulae

(4.4) $$B^{i}_{\beta;\gamma} = H_{\beta\gamma} N^{i},$$

where

$B^{i}_{\beta;\gamma}$ is the relative covariant derivative of $B^{i}_{\beta}$ with respect to $\underline{\Gamma}$ i.e.

(4.5) $$\beta^{i}_{\beta;\gamma} = B^{i}_{\beta\gamma} + B^{j}_{\beta} F^{i}_{j\gamma} - B^{i}_{\alpha} \Gamma^{\alpha}_{\beta\gamma}, \qquad B^{i}_{\beta\gamma} = \frac{\partial^{2} x^{i}}{\partial u^{\beta} \partial u^{\gamma}},$$

and $H_{\beta\gamma}$ is the second fundamental tensor. The torsion tensor $T^{\alpha}_{\beta\gamma}$ of the $\underline{\Gamma}$ is given by

(4.6) $$T^{\alpha}_{\beta\gamma} = B^{\alpha}_{i}(B^{j}_{\beta} F^{i}_{j\gamma} - B^{j}_{\gamma} F^{i}_{j\beta}).$$

From (4.4) and (4.5) we get

(4.7) $$H_{\beta\gamma} - H_{\gamma\beta} = N_{i}(B^{j}_{\beta} F^{i}_{j\gamma} - B^{j}_{\gamma} F^{i}_{j\beta}).$$

To consider the relative covariant derivative of a tensor field of $F^{n}$ along $M^{n-1}$, we shall deal with a Finsler vector field $x^{i}$ (x,y) From (4.3)

we have

(4.8) $$\frac{\partial X^{i}(x, N)}{\partial u^{\alpha}} = \left\{ \frac{\delta X^{i}(x, N}{\delta x^{j}} \right\} B^{j}_{\alpha} + \left\{ \frac{\partial X^{i}(x, N)}{\partial N^{j}} \right\} N^{j}_{\alpha},$$

where $$\frac{\delta}{\delta x^{j}} = \frac{\partial}{\partial x^{j}} - \frac{\partial}{\partial N^{r}} N^{r}_{j}(x, N).$$

Therefore, in terms of h- and v- covariant derivatives in $F^n$, we get

(4.9) $$x^i_{;\alpha} = X^i{}_{|j}(x,N)\,B^j_\alpha + X^i|_j(x,N)\,N^j_\alpha ,$$

where

(4.10) $$x^i_{;\alpha} = \frac{\partial X^i(x,N)}{\partial u^\alpha} + X^h(x,N)\,F^i_{h\alpha} ,$$

is the relative covariant derivative of $X^i$.

From (4.2) and (4.10), the relative covariant derivative of $N^i$ is given by

(4.11) $$N^i_{;\alpha} = \frac{\partial N^i}{\partial u^\alpha} + N^i_j(x,N)\,B^j_\alpha ,$$

which is nothing but $N^i_\alpha(u)$ given by (4.3).

Now we shall find $\frac{\partial N^i(x,B)}{\partial x^j}$. Differentiating (3.4) with respect to $x^k$ and nothing that $\frac{\partial g_{ij}(x,p)}{\partial p^h}p^j$ vanishes,

we have

$$g_{ij}\frac{\partial p^j(x,B)}{\partial x^k} = -\frac{\partial g_{ij}}{\partial x^k}p^j .$$

Contracting it by $g^{ih}$ (x,p) , we get

$$\frac{\partial p^h}{\partial x^k} = -g^{ih}p^j\frac{\partial g_{ij}}{\partial x^k} = -g^{ih}p^j(2C_{ijr}N^r_k + F_{ijk} + F_{jik} + a_k\,g_{ij}),$$

which gives

(4.12) $$\frac{\partial p^h(x,B)}{\partial x^k} = -F^h_{.0k}(x,p) - N^h_k(x,p) - p^h a_{k,}$$

where 0 denotes the contraction with $y^i$,for instance $F^h_{ok} = F^h_{ik}\,y^i$ and ‘. ‘ indicates that the raising index is of that place i.e. $F^h_{jk} = g^{ih}F_{ijk}$.

Next we have

$$\frac{\partial L(x, p(x,B))}{\partial x^k} = \frac{\partial L(x,p)}{\partial p^h}\left\{ N^{\,h}_{\,k}(x, p + \frac{\partial p^h}{\partial x^k} \right\} + \frac{a_k\, g_{ij}\, p^i\, p^j}{2L(x,p)}.$$

Then the identity $\dfrac{\partial L(x, p(x,B))}{\partial p^h} = \dfrac{q_h}{L(x,p)}$ and (4.12) lead to

(4.13) $$\frac{\partial L(x, p(x,B))}{\partial x^k} = -\frac{N^o_k(x,p)}{L(x,p)} - \frac{1}{2} a_k\, L(x,p).$$

Differentiating (3.5) with respect to $x^k$ and using (4.12) and (4.13), we obtain

(4.14) $$\frac{\partial N^i(x,B)}{\partial x^j} = \left\{ -F_{*o\,j}^{\;\;i} - N^i_j + N^i N^o_j - \frac{1}{2} N^i a_j \right\}_{y=N}.$$

Therefore (4.14) and (3.10) yield

$$\frac{\partial N^i}{\partial u^\alpha} - + N^{\,i}_{\,j}(x,N)\, B^j_\alpha = \left( -F_{*o\,j}^{\;\;i} + N^i N^{\,o}_{\,j} - \frac{1}{2} N^i a_j \right) B^{\,j}_{\,\alpha} - B^{\beta i} N_j\, B^j_{\beta\alpha},$$

and equations (4.14), (4.5) show that its right hand side is equal to

$$-H^\beta_\alpha\, B^{\,i}_{\,\beta} - \frac{1}{2} a_\alpha\, N^i,$$

where

$$H^{\,\beta}_{\,\alpha} = g^{\beta\gamma} H_{\gamma\alpha} \text{ and } a_\alpha = a_i\, B^i_\alpha.$$

As a consequence, we get the so- called Weingarten formulae

(4.15). $$N^{\,i}_{\,;\alpha} (= N^{\,i}_{\,\alpha}) = -H^{\,\beta}_{\,\alpha} B^{\,i}_{\,\beta} - \frac{1}{2} a_\alpha N^i.$$

## 5. Minimal Hypersurface

Consider the volume element $\sqrt{\underline{g}(u)}$, $\underline{g}(u) = \det(g_{\alpha\beta})$, where $g_{\alpha\beta}(u)$ is the metric tensor of the induced Riemannian N- metric (3.6) of the hypersurface $M^{n-1}$. If we put g (x, N) = $\det(g_{ij}(x, N))$ then (3.6), (3.1) and (3.2) give

$$\underline{g} = \begin{vmatrix} g_{ij}(x,N)\, B^i_\alpha B^j_\beta & g_{ij}(x,N)\, B^i_\alpha N^j \\ g_{ij}(x,N)\, N^i B^j_\beta & g_{ij}(x,N)\, N^i N^j \end{vmatrix} = g \left\{ \det(B^i_\alpha, N^i) \right\}^2$$

and (3.8) gives $\det(B^i_\alpha, N^i) = q_i N^i = L(x, p)$.

Thus we have

(5.1) $$\underline{g}(u) = g(x, N)\, L^2(x.p).$$

It is noted that, by the homogeneity property of $g_{ij}(x, y)$, $g(x, N)$ may be replaced by $g(x, p)$.

**Defination 5.1.** If the volume integral $I = \int \sqrt{\underline{g}(u)}\, du^1 du^2 .......du^{n-1}$ over a compact hypersurface $M^{n-1}$ has a stationary value i. e. vanishing first variation, $M^{n-1}$ is called a minimal hypersurface.

It is well known that the generalized Euler- Lagrange equation

(5.2) $$\frac{\partial \sqrt{\underline{g}}}{\partial x^i} - \frac{\partial}{\partial u^\alpha}\left( \frac{\partial \sqrt{\underline{g}}}{\partial B^i_\alpha} \right) = 0.$$

Characterizes minimal hypersrface. We have to write (5.2) in terms of quantities of $M^{n-1}$

The well-known equation $\partial g(x, y) / \partial y^i = 2g\, C_i, (C = C_{ijk}\, g^{ik})$ and (3.10) together give

(5.3) $$\frac{\partial g(x, p)}{\partial B^i_d} = -2g(x, N)\, C^\alpha(u)\, N_i .$$

where $C^\alpha(u) = C^i(x, N)\, B^\beta_i$ and $C^i = C_j\, g^{ij}$.

Then (5.3), (5.1) and (3.9) give

(5.4) $$\frac{\partial \sqrt{\underline{g}}}{\partial B^i_\alpha} = \sqrt{\underline{g}}\, (B^\alpha_i - C^\alpha N_i).$$

Next from (4.12) we have

(5.5) $$\frac{\partial g(x, p(x,B))}{\partial x^i} = \left( \frac{\delta g}{\delta x^i} - 2_g C_j F^{\ j}_{.*i} \right)_{y=p}.$$

Since $\frac{\delta g}{\delta x^i} = \frac{\delta g_{jk}}{\delta x^i} g\, g^{jk}$, therefore using the recurrence condition,

$g_{jk\backslash i} = a_i g_{jk,}$ We have

(5.6) $$\frac{\delta g}{\delta x^i} = g\,(2F^{\ j}_{ji} + na_i).$$

From (5.5) and (5.6) we have

(5.7) $$\frac{\partial g(x, p(x,B))}{\partial x^i} = g(x,n)[2F^{\ j}_{ji} + na_i - 2C_j F^{\ j}_{.*o\ i}]_{y=N}.$$

Now (5.1), (5.7) and (4.13) give

(5.8) $$\frac{\partial \sqrt{\underline{g}(u)}}{\partial x^i} = \sqrt{\underline{g}(u)}\,[F^{\ j}_{ij} - C_j F^{\ j}_{*oi} - N^0_{\ i} + \frac{1}{2}(n-1)a_i\,]_{y=N}.$$

Now to find $\frac{\partial}{\partial u^\alpha}\left( \frac{\partial \sqrt{\underline{g}}}{\partial B^i_\alpha} \right)$, we consider $\frac{\partial \sqrt{\underline{g}}}{\partial u^\alpha}, \frac{\partial B^\alpha_i}{\partial u^\alpha}, \frac{\partial C^\alpha}{\partial u^\alpha}$ and $\frac{\partial N_i}{\partial u^\alpha}$.

First we have,

$$\frac{\partial \sqrt{\underline{g}}}{\partial u^\alpha} = \frac{\partial \sqrt{\underline{g}}}{\partial x^i} B^{\ i}_{\alpha} + \frac{\partial \sqrt{\underline{g}}}{\partial B^{\ i}_{\beta}} B^{\ i}_{\beta\alpha},$$

then using (5.4), (5.8), (4.5) and (4.2) we have

(5.9) $$\frac{\partial \sqrt{\underline{g}}}{\partial u^\alpha} = \sqrt{\underline{g}} \left[ \Gamma^{\ \beta}_{\beta\alpha} + \frac{1}{2}(n-1)\, a_i\, B^i_\alpha \right].$$

Secondly (4.4) gives $B^{\ \alpha}_{i;\alpha} = MN_i$, where $M = g^{\alpha\beta} H_{\alpha\beta}$ is the mean curvature.

Therefore

(5.10) $$\frac{\partial B_i^\alpha}{\partial u^\alpha} = MN_i + B_j^\alpha F_{i\alpha}^j - B_i^\beta \Gamma_{\beta\alpha}^\alpha .$$

From (4.2) and (4.15), we have

(5.11) $$B_j^\alpha F_{i\alpha}^j = F_{ij}^j - F_{ioo} - B_i^N \varsigma^{\alpha\beta} H_{\alpha\beta} .$$

Then (5.10) (5.11) lead to

(5.12) $$\frac{\partial B_i^\alpha}{\partial u^\alpha} = MN_i + \left(F_{ij}^j - F_{i00}\right)_{y=N} - B_i^\gamma (C_\gamma^{\alpha\beta} H_{\alpha\beta} + \Gamma_{\gamma\alpha}^\alpha ).$$

It is well known that $C_{;\alpha}^\alpha = \frac{\partial C^\alpha}{\partial u^\alpha} + C^\beta \Gamma_{\beta\alpha}^\alpha$. Therefore equations (4.9), (4.4), (4.5) and condition $C^i{}_{|j} N^j = 0$ give

$$C^\alpha{}_{;\alpha} = (C^i B_i^\alpha)_{;\alpha} = C^i{}_{|i} (x, N) - C^i|_j (x, N) B_\beta^j H_\alpha^\beta B_i^\alpha - \frac{1}{2} C^i|_j a_\alpha N^j B_i^\alpha .$$

Hence

(5.13) $$\frac{\partial C^\alpha}{\partial u^\alpha} = C_{|i}^i (x, N) - C^i|_j (x, N) B_\beta^j H_\alpha^\beta B_i^\alpha - \frac{1}{2} C^i|_j a_\alpha N^j B_i^\alpha - C^\beta \Gamma_{\beta\alpha}^\alpha .$$

From (4.5) we have $N_{i;\alpha} = -H_\alpha^\beta B_{\beta i} + \frac{1}{2} a_\alpha N_i$.

Hence

(5.14) $$\frac{\partial N_i}{\partial u^\alpha} = F_{i0j} (x, N) B_\alpha^j - H_{\beta\alpha} B_i^\beta + \frac{1}{2} a_\alpha N_i .$$

Now from (5.4), (5.9), (5.12) and (5.14), we have

(5.15) $$\frac{\partial}{\partial u^\alpha}\left\{\frac{\partial \sqrt{g}}{\partial B_\alpha^i}\right\} = \sqrt{\underline{g}}\,[B_i^\alpha \left\{T_{\beta\alpha}^\beta - C_\alpha^{\beta\gamma} H_{\beta\gamma} + C^\beta H_{\alpha\beta}\right\} + N_i \{M + C^j|_k B_\beta^k H_\alpha^\beta B_j^\alpha ,$$
$$+ \frac{1}{2} C^j|_k N^k a_\alpha B_j^\alpha - C^j|_j - T_{\beta\alpha}^\beta C^\alpha - \frac{1}{2}(n-1)\left(a_0 + \frac{n}{n-1} a_j C^j\right)\Big\},$$
$$+ (F_{ij}^j - F_{ioo} - F_{ioj} C^j) + \frac{1}{2}(n-1) a_i]\ y = N.$$

Substituting the value of $\dfrac{\partial\sqrt{\underline{g}(u)}}{\partial x^i}$ from (5.8) and that of $\dfrac{\partial}{\partial u^\alpha}\left\{\dfrac{\partial\sqrt{\underline{g}}}{\partial B^i_\alpha}\right\}$

from (5.15) in (5.2), we have

$$(5.16)\quad \sqrt{\underline{g}}\Big[T^{\,j}_{ji}+T_{i0j}\,C^j+T_{i00}-B^\alpha_i\,(T^\beta_{\beta\alpha}-C^{\beta\gamma}_\alpha\,H_{\beta\gamma}+C^\beta\,H_{\alpha\beta}), \\ -N_i, \\ -\frac{1}{2}(n-1)(a_0+\frac{n}{n-1}a_j\,C^j)\Big\}\Big]_{y=N}=0.$$

We quote the following results, which may be derived by equations in section 4.

$$(5.17)\quad \begin{bmatrix} T^\beta_{\beta\alpha}\,B^\alpha_i=T^{\,j}_{ji}+T_{i00}-N_i\,T^{\,j}_{j0}+B^\alpha_i\left(C^{\beta\gamma}_\alpha\,H_{\beta\gamma}\,C^\beta\,H_{\beta\alpha}\right)-\frac{1}{2}C_k\,N^k\,a_\alpha\,B^\alpha_i, \\ B^\alpha_i\,H_{\alpha\beta}\,C^\beta=C^\beta\,H_{\beta\alpha}\,B^\alpha_i+T_{i0j}\,C^j-N_i\,T_{00j}\,C^j, \\ C^\alpha\,T^\beta_{\beta\alpha}=T^{\ \,j}_{ji}\,C^i-H_{\alpha\beta}\,C^\alpha\,C^\beta-T_{00i}\,C^i+C^\gamma\,C^{\alpha\beta}_\gamma\,H_{\alpha\beta}. \end{bmatrix}$$

Also the definition of $C^j|_k$ shows that

$$(5.18)\quad \begin{bmatrix} C^j|_k\,B^k_\beta\,H^\beta_\alpha\,B^\alpha_j=(\dot\partial_k\,C^j)\,H^\beta_\alpha\,B^\alpha_j\,B^k_\beta+C^\gamma\,C^{\alpha\beta}_\gamma\,H_{\alpha\beta}, \\ C^j|_k\,a_\alpha\,N^k\,B^\alpha_j=(\dot\partial_k\,C^j)\,a_\alpha\,N^k\,B^\alpha_j. \end{bmatrix}$$

On using (5.17) and (5.18) in (5.16) we see that the resulting equation has only normal components. Hence, in scalar form, this equation reads as

$$(5.19)\quad \left\{T^i_{i0}+T^i_{ij}\,C^j+C^i|_i+\frac{1}{2}(n-1)(a_0+\frac{n}{n-1}a_j\,C^j)\right\}_{y=n} \\ =H_{\alpha\beta}\,B^\alpha_i\,B^\beta_j\,(g^{ij}+C^i\,C^j+g^{ik}\,\dot\partial_k\,C^j)+\frac{1}{2}(\dot\partial_j\,C^i)\,N^j\,a_\alpha\,B^\alpha_i.$$

This very equation characterizes minimal hypersurface with respect to recurrent generalized Cartan connection Rec $C\Gamma(a,T)$.

If we are concerned with generalized Cartan connection (a=0) , (5.19) becomes

(5.20) $$T^i_{i0} + T^i_{ij}\, C^j + C^i|_i = H_{\alpha\beta}\, B^\alpha_i\, B^\beta_j\, (g^{ij} + C^i\, C^j + g^{ik}\, \dot{\partial}_k\, C^j),$$

which is the characterizing equation for a minimal hypersurface with respect to generalized Cartan connection $C\Gamma(T)$.

Therefore we have

**Theorem5.1.** *A hyperplane* $(H_{\alpha\beta}=0)$ *is minimal with respect to generalized Cartan connection if and only if the ambient space* $F^n$ *satisfies the condition*

$$T^{\,i}_{\,io} + T^{\,i}_{ij}\, C^j + C^{\,i}|_i = 0.$$

*If we are concered with Cartan connection (a=0, T=0) , (5.19) becomes*

*(5.21)* $$C^{\,i}_{\ |i}\,(x,N) = H_{\alpha\beta}\, B^\alpha_i\, B^\beta_j\, (g^{ij} + C^i\, C^j + g^{ik}\, \dot{\partial}_k\, C^j).$$

Hence we have[6]

**Theorem5.2.** *A hyperplane* $(H_{\alpha\beta}=0)$ *is minimal with respect to Cartan connection if and only if* $C^{\,i}_{\ \ |i}=0$.

*If we are concerned with recurrent Barthel's connection for which*

$$T^i_{ik} = L(1_j\, C^{\,i}_{\ |k} - 1_k\, C^i_{\ \ |j}),$$

*we have*

$$T^i_{ik} = (x,N) = -N_k\, C^{\,i}_{\ |i}\,(x,N),\ and T^i_{ik} = (x,N)\, C^k\,(x,N) = 0,$$

*then (5.19) reduces to*

(5.22) $$\frac{1}{2}(n-1)(a_0 + \frac{n}{n-1}(a_j\, C^j)\, H_{\alpha\beta}\, B^\alpha_i\, B^\beta_j\, (g^{ij} + C^i\, C^j + g^{ik}\, \dot{\partial}_k\, C^j)_{y=N},$$
$$+\frac{1}{2}\dot{\partial}_j\, C^i\, N^j\, a_\alpha\, B^\alpha_i.$$

As a consequence, we have the following:

**Theorem 5.3.** *A hyperplane* $(H_{\alpha\beta}=0)$ *is minimal with respect to recurrent Barthel connection if and only if the ambient space* $F^n$ *satisfies the condition*

$$\frac{1}{2}(n-1)(a_0+\frac{n}{n-1}a_j\,C^j)=\frac{1}{2}\dot{\partial}_j\,C^i\,N^j\,a_\alpha\,B_i^\alpha\,.$$

*For h-metrical* $(a=0)$ *Barthel connection, (5.22) redues to*

(5.23) $$H_{\alpha\beta}\,B_i^\alpha\,B_j^\beta\,(g^{ij}\,C^i\,C^j+g^{ik}\,\dot{\partial}_k\,C^{j)}\,{}_{y=n}=0.$$

Hence we have the following[6].

**Theorem 5.4.** *If we are concerned with metrical Barthel connection, a hyperplane is necessarily minimal.*

## References

1. E. Cartan, *Les espaces de Finsler,* Actualities, **79** Paris 1934.
2. M. Matsumoto**,** The induced and intrinsic Finsleronnections of a hypersurface and Finslerian projective geometry. *J. Math, Kyoto Univ*., **25** (1985) 107-144.
3. J. M. Wegener, Untersuchungenuber Finslerschen Raume, *LotosPrag*, **84** (1936) 4-7.
4. J. M. Wegener, Hyperflachen in Finslershen Raumenals Transversal facheneinerschar von Extremalen, *Monatsh. Fur Math. And Phy*., **44** (1936) 115-130.
5. E. T. Davies, Subspaces of a Finsler space, *Proc. Lond. Math. Soc.,* **49(2)** (1947) 19-39.
6. M. Matsumoto, Theory of y-extremal and minimal hypersurface in a Finsler space, *J. Math. Kyoto Univ*., **26(4)** (1986) 647-665.
7. B. N. Prasad, H. S. Shukla and D. D. Singh, On recurrent connections with deflection and torsion, *Publicationes Mathematicae*, Debrecen **37** (1990) 77-84.
8. H. S. Shukla, B. N. Prasad and A. P. Tiwari, *Y-extremalhypersurface of a Finsler space*, Communicated.
9. M. Matsumoto, A Finsler connection with many torsions, *Tensor, N. S*., **17** (1966) 217-226.
10. M. Matsumoto, *The Theory of Finsler Connections*, Publ. study group of geometry, **5** Dept. Math. Okayama Univ., 1970.