ISSN 0974 - 9373

Vol. 16 No.4 (2012) **Journal of International Academy of Physical Sciences** pp. 343-361

# Surface Waves in Prestressed Elastic Multilayered Media

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(Received December 22, 2011)

**Abstract:** Thomson-Haskell matrix method is used to study the propagation of Rayleigh waves in prestressed multilayered elastic solid with incremental elastic coefficients possessing orthotropic symmetry. The dispersion equations are derived for one and two prestressed layered half space.
**Keywords:** Surface wave, multilayered media, prestessed media, dispersion equation.

## 1. Introduction

In fact the velocity of waves depends on the material properties (elastic moduli and density) allows us to use seismic wave observations to investigate the interior structure of the planet. We can look at the travel times and the amplitudes of waves to infer the existence of features within the planet, and this is an active area of seismological research. To understand how we “see” into Earth using vibrations, we must study how waves interact with the rocks that make up Earth. Surface waves carry the greatest amount of energy from shallow shocks and are of primary cause of destruction that can result from earthquakes .Surface waves propagating over the surface of homogeneous and inhomogeneous elastic half space are a well- known and prominent features of waves theory.

The Early attempts were made, to drive the dispersion equations for surface waves of Rayleigh types on a layered medium by Sezawa[1], Fu[2] and others. But the methods used by them, were so formidable that no attempt appears to have been made to treat cases of more than two layers. A straight forward matrix method was investigated by Thomson[3]. Haskell (1953) followed the same technique to study the dispersion equations for Love and

Rayleigh waves in homogeneous isotropic media .This method has been adopted by several investigators e.g. Anderson[4],Hannon[5] and Saini[6].
All these study ignore the initial stresses present in the medium. Adopting the basic theory of Biot[7] on prestressed solids in last three decades ,the propagation of elastic waves in prestressed solids of infinite extent has been discussed by Dahlen[8],Walton[9],Tolstoy[10], Dey and Chakraborty[11], Sidhu and Singh[12], Singh and Singh[13].

Recently Addy and Chakraborty[14] showed the effect of temperature and initial hydrostatic stress on the propagation of Rayleigh waves in viscoelastic half space .Sharma and Garg[15] have derived Modified Christoffel equations for three dimensional wave propagation in general anisotropic medium under initial stress. Gupta et al.[16] found out the effect of initial stress on propagation of Love waves in an anisotropic porous layer.

The model of the Earth is supposed to be composed of several layers of different thickness and inner layers of the Earth are under immense stress owing to different physical causes i.e. variation in temperature ,slow process of creep and gravitational field. Therefore, the study of propagation of elastic waves in the prestressed elastic multilayered media is of great importance in seismology.

Keeping in view, the above aspects, study of propagation of Rayleigh waves in layered prestressed elastic media have been dealt in the present paper following Thomson-Haskell matrix method. Dispersion equations of Rayleigh waves are derived for one and two layers respectively .The results obtained in this analysis are compared with those for an initially stress free medium.

## 2. Basic Theory

The general form of Biot's field equations for prestressed solids in the absence of external forces is (Tolstoy[10])

(2.1) $$s_{ij,j} + S_{jk}\omega_{ik,j} + S_{ik}\omega_{jk,j} - e_{jk}S_{ik,j} = \rho u_{i,tt}$$

Here $u_i$ are the displacement components, ρ is density and

(2.2) $$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad \omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}),$$

$$u_{i,j} = \frac{\partial u_i}{\partial x_j},\ u_{i,t} = \frac{\partial u_i}{\partial t}, \qquad etc.$$

The $S_{ij}$ are the components of prestress which are assumed to be satisfy the equilibrium equations

(2.3) $$S_{ij,j} = 0.$$

The $s_{ij}$ are the incremental stresses and are assumed to be linearly related to the incremental strains $e_{ij}$:

$$s_{11} = B_{11}u_x + B_{12}(v_y + w_z),$$
$$s_{22} = \left(B_{12} - P\right)u_x + B_{22}v_y + B_{23}w_z,$$
$$(2.4) \quad s_{33} = \left(B_{12} - P\right)u_x + B_{23}v_y + B_{22}w_z,$$
$$s_{12} = Q_2\left(u_y + v_x\right),\ s_{13} = Q_2\left(u_z + w_x\right),$$
$$s_{23} = Q_1\left(v_z + w_y\right),$$

where we have put

$$(x_1, x_2, x_3) = (x, y, z),$$
$$(2.5) \quad (u_1, u_2, u_3) = (u, v, w),\ u_x = \frac{\partial u}{\partial x}, etc.$$

Further

$$B_{11} = \left(2\mu + \lambda\right)\left(1 + \varepsilon_{11} - 2\varepsilon_{22}\right),$$
$$B_{22} = \left(2\mu + \lambda\right)\left(1 - \varepsilon_{11}\right),$$
$$B_{12} = \lambda\left(1 - \varepsilon_{22}\right) - S_{11},$$
$$(2.6) \quad B_{23} = \lambda\left(1 - \varepsilon_{11}\right) - S_{23,}$$

$$P = S_{33} - S_{11}$$

$$Q_1 = \mu + \left(\mu + \lambda\right)\varepsilon_{22} + \frac{1}{2}\left(\lambda - 2\mu\right)\varepsilon_{11},$$

$$Q_1 = \mu + \frac{1}{2}\left(\mu + \lambda\right)\left(\varepsilon_{11} + \varepsilon_{22}\right) + \frac{1}{2}\left(\lambda - 2\mu\right)\varepsilon_{22}$$

In deriving (2.6), we have taken $S_{22} = S_{33}$ .Futher, $\lambda, \mu$ are Lame's constants and $\varepsilon_{ij}$ are the initial strains.

We now consider the two dimensional problem for which $\frac{\partial}{\partial y} \equiv 0$ .We further assume that

$$(2.7) \qquad S_{12} = S_{13} = S_{23} = 0,$$

and $S_{11}, S_{33}$ are constants. Equation (2.2) shows that

(2.8) $$e_{11} = u_x, e_{33} = w_z,\ e_{13} = \frac{1}{2}\left(u_z + w_x\right),$$

$$e_{12} = \frac{1}{2}v_x, e_{23} = \frac{1}{2}v_z, e_{22} = 0.$$

(2.9) $$w_{23} = \frac{1}{2}v, \omega_{31}$$

$$\omega_{11} = \omega_{22} = \omega_{33} = 0.$$

On taking i=1,2,3 in equation (2.1) and using equations (2.4), (2.7) - (2.9), we get

$$B_{11}\frac{\partial^2 u}{\partial x^2} + \left(B_{12} + Q_2 - \frac{1}{2}P\right)\frac{\partial^2 w}{\partial x \partial z} + \left(Q_2 + \frac{1}{2}P\right)\frac{\partial^2 u}{\partial z^2},$$

(2.10) $$\left(Q_2 - \frac{1}{2}P\right)\frac{\partial^2 v}{\partial x^2} + Q_1\frac{\partial^2 v}{\partial z^2} = \rho\frac{\partial^2 v}{\partial t^2},$$

$$B_{22}\frac{\partial^2 w}{\partial z^2} + \left(B_{12} + Q_2 - \frac{1}{2}P\right)\frac{\partial^2 u}{\partial x \partial z} + \left(Q_2 - \frac{1}{2}P\right)\frac{\partial^2 w}{\partial x} = \rho\frac{\partial^2 w}{\partial t^2}.$$

If we restrict our analysis to xz- plane ,put v =0 from first and last member of equation (2.10), we get

$$B_{11}\frac{\partial^2 u}{\partial x^2} + A_3\frac{\partial^2 w}{\partial x \partial z} + A_1\frac{\partial^2 u}{\partial z^2} = \rho\frac{\partial^2 u}{\partial t^2},$$

(2.11)

$$B_{22}\frac{\partial^2 w}{\partial z^2} + A_3\frac{\partial^2 u}{\partial x \partial z} + A_2\frac{\partial^2 w}{\partial x} = \rho\frac{\partial^2 w}{\partial t^2},$$

where $A_1 = \left(Q_2 + \frac{1}{2}P\right), A_2 = \left(Q_2 - \frac{1}{2}P\right)$ and $A_3 = \left(B_{12} + Q_2 - \frac{1}{2}P\right).$

We assume the solution of equation (2.11) in the form as

$$u = U_i e^{(iP_i)},$$

$$(2.12) \qquad w = W_i e^{(iP_i)}$$

where $U_i$ and $W_i$ are the amplitude factors and

$$(2.13) \qquad P_1 = k\left\{ct - \left(x\sin\theta - z\cos\theta\right)\right\},$$ where

$k =$ wave number,
$c =$ phase velocity,
$P_1 =$ Phase factor.
Substituting equations (2.12) & (2.13) in equation (2.11), we have

$$(2.14) \qquad -\left(D_1 - \rho c^2\right)U_i + A_3 \sin\theta\cos\theta W_i,$$

$$A_3 \sin\theta\cos\theta U_i - \left(D_2 - \rho c^2\right)W_i,$$

where

$$(2.15) \quad D_1\left(\theta\right) = B_{11}\sin^2\theta + A_1\cos^2\theta,$$

$$D_2\left(\theta\right) = B_{22}\cos^2\theta + A_2\sin^2\theta.$$

The set of homogeneous equations (2.14) in $U_i, W_i$ has a non-trivial solution only if $\begin{vmatrix} -\left(D_1 - \rho c^2\right) & A_3\sin\theta\cos\theta \\ A_3\sin\theta\cos\theta & -\left(D_2 - \rho c^2\right) \end{vmatrix} = 0$.

It is biquadratic in $c$,

$$(2.16) \qquad 2\rho c^2\left(\theta\right) = \left(D_1 + D_2\right) \mp \left[\left(D_1 - D_2\right)^2 + 4A_3^{\,2}\sin^2\theta\cos^2\theta\right]^{1/2}$$

Thus, in general, in this two dimensional model of the prestressed solid, there are two types of plane waves i.e. quasi P waves and quasi SV waves whose velocities depend on the initial stresses and direction of propagation .

## 3. Formulation of The Problem

Consider a homogeneous prestressed elastic half-space composed of n parallel ,homogeneous, prestressed plane layers. The nth layer being a homogeneous, prestressed elastic half-space, various layers and interfaces are numbered away from free surface, as shown in (Fig.1). Let the material is either isotropic in finite strain or anisotropic with orthotropic symmetry. The principal directions of initial stress are chosen to coincide with the directions of elastic symmetry and the co-ordinate axes. The state of initial stress is, therefore, defined by principal components $S_{11}, S_{22}$ and $S_{33}$ of the initial stress. We assume plane strain parallel to the xz- plane and z-axis is taken as directed into the medium. The equations of motion for $m^{th}$ layer are given as in equation (3.1)

$$(3.1) \qquad B_{11,m}\frac{\partial^2 u_m}{\partial x^2}+A_{3,m}\frac{\partial^2 w_m}{\partial x \partial z}+A_{1,m}\frac{\partial^2 u_m}{\partial z^2}=\rho_m\frac{\partial^2 u_m}{\partial t^2},$$

$$B_{22,m}\frac{\partial^2 w_m}{\partial x^2}+A_{3,m}\frac{\partial^2 u_m}{\partial z \partial x}+A_{2,m}\frac{\partial^2 w_m}{\partial x^2}=\rho_m\frac{\partial^2 w_m}{\partial t^2}.$$

where $\rho_m$ is the density and $(u_m, w_m)$ are displacement components in x and z directions, respectively and

$$(3.2) \qquad A_{1,m}=Q_{2,m}+\frac{P_m}{2}, A_{2,m}=Q_{2,m}-\frac{P_m}{2},$$

$$A_{3,m}=B_{12,m}+Q_{2,m}-\frac{P_m}{2}, P_m=S_{33,m}-S_{11,m}.$$

$B_{11,m}$, $B_{22,m}$ are the incremental elastic coefficients in the mth layer. When the initial state is unstressed, the medium is isotropic, and the first order theory of classical elasticity is assumed, them it can be shown that

$$(3.3) \qquad \begin{aligned} &B_{11,m}=B_{22,m}=\lambda_m+2\mu_m \\ &B_{12,m}=\lambda_m, \\ &Q_{2,m}=\mu_m, A_{1,m}=A_{2,m}=\mu_m, \\ &A_{3,m}=\lambda_m+\mu_m, \end{aligned}$$

where $\lambda_m$ *and* $\mu_m$ are the usual Lame parameters.

For plane waves of circular frequency $\omega$, wavenumber $k$ and phase velocity $c$ ,propagating in the x-direction, are incident at the free boundary z=0 of a semi-infinite prestressed medium, both quasi-P and quasi-SV waves will be generated. we may assume the total displacement field to be of the form

$$(3.4) \qquad \begin{aligned} u_m = {} & U_{i1,m}\exp(lP_{1,m})+U_{i2,m}\exp(lP_{2,m}) \\ & +U_{r1,m}\exp(lR_{1,m})+U_{r2,m}\exp(lR_{2,m}), \end{aligned}$$

where

$$w_m = W_{i1,m} \exp\left(lP_{1,m}\right) + W_{i2,m} \exp\left(il\right)$$
$$+ W_{r1,m} \exp\left(lR_{1,m}\right) + W_{r2,m} \exp\left(lR_{2,m}\right),$$

(3.5)
$$P_{1,m} = P_{1,m}(x,z) = \frac{w}{c_{1,m}}\left\{C_{1,m}t - \left(x\sin e_m - z\cos e_m\right)\right\},$$

$$P_{2,m} = P_{2,m}(x,z) = \frac{w}{c_{2,m}}\left\{C_{2,m}t - \left(x\sin f_m - z\cos f_m\right)\right\},$$

$$R_{1,m} = R_{1,m}(x,z) = \frac{w}{c_{1,m}}\left\{C_{1,m}t - \left(x\sin e_m + z\cos e_m\right)\right\},$$

$$R_{2,m} = R_{2,m}(x,z) = \frac{w}{c_{2,m}}\left\{C_{2,m}t - \left(x\sin f_m + z\cos f_m\right)\right\},$$

$P_{1,m}$ and $R_{1,m}$ are the phase factors associated with incident and the reflected quasi-P waves, respectively, $e_m$ being the angle which these waves make with z-axis.$P_{2,m}$ and $R_{2,m}$ are the phase factors associated with the incident and the reflected quasi-S waves, respectively, $f_m$ beings the angle which these waves make with z-axis.$\left(U_{i1,m}, W_{i1,m}\right)$ and $(U_{r1,m}, W_{r1,m})$ are the amplitude factors associated with the incident and reflected P-waves, $(U_{i2,m} + W_{i2,m})$ *and* $(U_{r2,m} + W_{r2,m})$ are amplitude factors associated with the incident and the reflected S-waves.

We find that the displacements given in equation (3.4) must satisfy the equation of motion (3.1), we have

(3.6)
$$U_{i1,m} = F_{1,m}W_{i1,m},\ U_{i2,m} = F_{2,m}W_{i2,m},$$
$$U_{i1,m} = -F_{1,m}W_{i1,m},\ U_{i1,m} = -F_{2,m}W_{i2,m},$$

*here*

$$F_{1,m} = \frac{A_{3,m}\sin e_m \cos e_m}{D_{1,m}\left(e_m\right) - \rho_m c_{1,m}{}^2\left(e_m\right)},$$

(3.7)
$$F_{2--,m} = \frac{A_{3,m}\sin f_m \cos f_m}{D_{1,m}\left(f_m\right) - \rho_m c_{2,m}{}^2\left(f_m\right)}.$$

## 4. Boundary Conditions

Let plane waves be incident from the prestressed isotropic elastic half-space at the (n-1)th interface at an angle $\theta_n$. Let the plane of incident be taken as xz-plane, we solve the problem subject to the following conditions:

(1) The free surface is stress free, that is,

(4.1) $\Delta f_x = 0,\ \Delta f_z = 0,\quad at\ z = 0. \Delta f_x = 0.$

(2) The displacements and incremental boundary forces are continuous at an interface. This implies

$$
\begin{aligned}
(4.2) \qquad & u_i(z_i) = u_{i+1}, \\
& w_i(z_i) = w_{i+1}, \\
& (\Delta f_x)_i(z_i) = (\Delta f_x)_{i+1}(z_i), \\
& (\Delta f_z)_i(z_i) = (\Delta f_z)_{i+1}(z_i), \\
& i = 1, 2, 3, \ldots\ldots, (n+1).
\end{aligned}
$$

$$
\begin{aligned}
& (\Delta f_x)_i = s_{13,i} + S_{33,i}(w_y)_i - S_{11,i}(e_{zx})_i, \\
(4.3) \qquad & (\Delta f_z)_i = s_{33,i} + S_{33,i}(e_{xx})_i.
\end{aligned}
$$

are the incremental boundary forces per unit initial area. $s_{13,i}$ *and* $s_{33,i}$ are incremental stresses, $e_{zx,i}, e_{xx,i}$ are incremental strains and $(w_y)_i$ is the incremental rotation component parallel to xz- plane. Explicit expressions for these quantities in terms of $u_i$ *and* $w_i$ are

$$
s_{13,i} = Q_{2,i}\left(\frac{\partial u_i}{\partial z} + \frac{\partial w_i}{\partial x}\right),
$$

$$
s_{33,i} = \left(B_{12,i} - P_i\right)\frac{\partial u_i}{\partial z} + B_{22,i}\frac{\partial w_i}{\partial z},
$$

$$
(4.4) \qquad (e_{xx})_i = \frac{1}{2}\left(\frac{\partial u_i}{\partial z} + \frac{\partial w_i}{\partial x}\right),
$$

$$
(w_y)_i = \frac{1}{2}\left(\frac{\partial u_i}{\partial z} + \frac{\partial w_i}{\partial x}\right).
$$

We temporarily shift the origin to the (m-1)th interface as shown in (Fig.2). Then, at the (m-1) th interface we have $z = 0$. We denote displacements and incremental stresses at the mth interface by

$u_m, w_m, (\Delta f_x)_m$ *and* $(\Delta f_z)_m$ respectively. Then for (m-1) th interface equation (3.4) together with the boundary condition (4.2) and equations (4.3) - (4.4), give

(4.5) $u_{m-1} = U_{i1,m} e^{iR_{1,m}(x,0)} + U_{i2,m} e^{iR_{2,m}(x,0)} + U_{r1,m} e^{iR_{1,m}(x,0)} + U_{r2,m} e^{iR_{2,m}(x,0)}$,

$$w_{m-1} = W_{i1,m} e^{iR_{1,m}(x,0)} + W_{i2,m} e^{iR_{2,m}(x,0)} + W_{r1,m} e^{iR_{1,m}(x,0)} + W_{r2,m} e^{iR_{2,m}(x,0)}$$

$$(\Delta f_x)_{m-1} = \frac{1}{c_{1,m}} \left\{ \begin{array}{l} (2Q_{2,m} + P_m)(U_{r1,m} - U_{i1,m}) \cos e_m \\ +(2Q_{2,m} - R_m)(W_{r1,m} - W_{i1,m}) \sin e_m \end{array} \right\} e^{iR_{1,m}(x,0)} +$$

$$\frac{1}{c_{2,m}} \left\{ \begin{array}{l} (2Q_{2,m} + P_m)(U_{r2,m} - U_{i2,m}) \cos f_m \\ +(2Q_{2,m} - R_m)(W_{r2,m} - W_{i2,m}) \sin f_m \end{array} \right\} e^{iR_{2,m}(x,0)},$$

$$(\Delta f_z)_{m-1} = \frac{1}{c_{1,m}} \left\{ \begin{array}{l} (B_{12,m} + S_{11,m})(U_{r1,m} - U_{i1,m}) \sin e_m \\ +(B_{22,m})(W_{r1,m} - W_{i1,m}) \cos e_m \end{array} \right\} e^{iR_{1,m}(x,0)} +$$

$$\frac{1}{c_{2,m}} \left\{ \begin{array}{l} (B_{12,m} + S_{11,m})(U_{r2,m} - U_{i2,m}) \sin f_m \\ +(B_{22,m})(W_{r2,m} - W_{i2,m}) \cos f_m \end{array} \right\} e^{iR_{2,m}(x,0)}.$$

*where*

(4.6) $c_{1,m} = c_1(e_m), \;\; c_{2,m} = c_2(e_m), \; R_m = S_{11,m} + S_{33,m},$

and we have made use of the results

$$P_{1,m}(x,0) = R_{1,m}(x,0),$$

(4.7) $P_{2,m}(x,0) = R_{2,m}(x,0).$

Since equation (4.5) must be satisfied for all values of x, we have

(4.8) $R_{1,m}(x,0) = R_{2,m}(x,0)$

Which, on using equation (3.2) implies

(4.9) $\dfrac{\sin e_m}{c_{1,m}(e_m)} = \dfrac{\sin f_m}{c_{1,m}(f_m)}.$

This is the form of Snell's law for initially stressed media. Using equations (3.5), (4.8) in equation (4.5) and suppressing the common factor

$$e^{\frac{i\omega}{c_{2,m}}\left[c_{2,m}t - x \sin f_m\right]},$$

*we get*

$$u_{m-1} = F_{1,m}W_{i1,m} + F_{2,m}W_{i2,m} - F_{1,m}W_{r1,m} - F_{2,m}W_{r2,m},$$

(4.10) $$w_{m-1} = W_{i1,m} + W_{i2,m} + W_{r1,m} + W_{r2,m},$$

$$\left(\Delta f_x\right)_{m-1} = a_{1,m}W_{i1,m} + a_{2,m}W_{i2,m} + a_{1,m}W_{r1,m} + a_{2,m}W_{r2,m},$$

$$\left(\Delta f_z\right)_{m-1} = b_{1,m}W_{i1,m} + b_{2,m}W_{i2,m} - b_{1,m}W_{r1,m} - b_{2,m}W_{r2,m},$$

Where

(4.11) $$c_{1,m}a_{1,m} = -\left(P_m + 2Q_{2,m}\right)F_{1,m}\cos e_m + \left(2Q_{2,m} - R_m\right)F_{1,m}\sin e_m,$$

$$c_{2,m}a_{2,m} = -\left(P_m + 2Q_{2,m}\right)F_{2,m}\cos f_m + \left(2Q_{2,m} - R_m\right)F_{1,m}\sin f_m,$$

$$c_{1,m}b_{1,m} = \left(B_{12,m} + S_{11,m}\right)F_{1,m}\sin e_m - B_{22,m}\cos e_m,$$

$$c_{2,m}b_{2,m} = \left(B_{12,m} + S_{11,m}\right)F_{2,m}\sin f_m - B_{22,m}\cos f_m.$$

Equation (4.10) can be written in te matrix form as

(4.12) $$\begin{bmatrix} u_{m-1} \\ w_{m-1} \\ \left(\Delta f_x\right)_{m-1} \\ \left(\Delta f_x\right)_{m-1} \end{bmatrix} = E_m \begin{bmatrix} W_{r1,m} + W_{i1,m} \\ W_{r1,m} - W_{i1,m} \\ w_{r2,m} - w_{i2,m} \\ w_{r2,m} + w_{i2,m} \end{bmatrix}$$

here

(4.13) $$E_m = \begin{bmatrix} 0 & -F_{1,m} & -F_{2,m} & 0 \\ 1 & 0 & 0 & 1 \\ a_{1,m} & 0 & 0 & a_{2,m} \\ 0 & -b_{1,m} & -b_{2,m} & 0 \end{bmatrix}$$

Similarly, from equation (3.1).We find the values of the displacements and stresses at the m$^{\text{th}}$ interface by putting $z = dm$

(4.14)
$$\begin{bmatrix} u_m \\ w_m \\ (\Delta f_x)_m \\ (\Delta f_z)_{\partial m} \end{bmatrix} = E_m \begin{bmatrix} W_{r1,m} + W_{i1,m} \\ W_{r1,m} - W_{i1,m} \\ w_{r2,m} - w_{i2,m} \\ w_{r2,m} + w_{i2,m} \end{bmatrix}$$

*where*

$$D_m = \begin{bmatrix} lF_{1,m}\sin\theta_m & -F_{1,m}\cos\theta_m & -F_{2,m}\cos\phi_m & F_{2,m}\sin\phi_m \\ \cos\theta_m & -l\sin\theta_m & -l\sin\phi_m & \cos\phi_m \\ a_{1,m}\cos\theta_m & a_{1,m}\sin\theta_m & -la_{2,m}\sin\phi_m & a_{2,m}\cos\phi_m \\ lb_{1,m}\sin\theta_m & -b_{1,m}\cos\theta_m & -lb_{2,m}\cos\phi_m & lb_{2,m}\sin\phi_m \end{bmatrix},$$

*Here*

(4.15) $$\theta_m = \frac{\omega dm \cos e_m}{c_{1,m}}$$

(4.16) $$\phi_m = \frac{\omega dm \cos f_m}{c_{2,m}}$$

(4.17)
$$\begin{bmatrix} W_{r1,m} + W_{i1,m} \\ W_{r1,m} - W_{i1,m} \\ W_{r2,m} - W_{i2,m} \\ W_{r2,m} + W_{i2,m} \end{bmatrix} = E_m^{-1} \begin{bmatrix} u_{m-1} \\ w_{m-1} \\ (\Delta f_x)_{m-1} \\ (\Delta f_x)_{m-1} \end{bmatrix},$$

a combination of equations (4.14) and (4.17), yields,

(4.18)
$$\begin{bmatrix} u_m \\ w_m \\ (\Delta f_x)_m \\ (\Delta f_z)_m \end{bmatrix} = D_m E^{-1}{}_m \begin{bmatrix} u_{m-1} \\ w_{m-1} \\ (\Delta f_x)_{m-1} \\ (\Delta f_x)_{m-1} \end{bmatrix},$$

where, from (4.13)

$$(4.19)\qquad E^{-1}{}_m = \begin{bmatrix} 0 & \dfrac{-a_{2,m}}{a_{2,m}-a_{1,m}} & \dfrac{1}{a_{2,m}-a_{1,m}} & 0 \\ \dfrac{b_{2,m}}{F_{1,m}b_{2,m}-F_{2,m}b_{1,m}} & 0 & 0 & \dfrac{-F_{2,m}}{F_{1,m}b_{2,m}-F_{2,m}b_{1,m}} \\ \dfrac{-b_{1,m}}{F_{1,m}b_{2,m}-F_{2,m}b_{1,m}} & 0 & 0 & \dfrac{-F_{1,m}}{F_{1,m}b_{2,m}-F_{2,m}b_{1,m}} \\ 0 & \dfrac{a_{1,m}}{a_{2,m}-a_{1,m}} & \dfrac{-1}{a_{2,m}-a_{1,m}} & 0 \end{bmatrix}$$

Equation (4.18) can be written as

$$(4.20)\qquad \begin{bmatrix} u_m \\ w_m \\ (\Delta f_x)_m \\ (\Delta f_z)_m \end{bmatrix} = A_m \begin{bmatrix} W_{r1,m}+W_{i1,m} \\ W_{r1,m}-W_{i1,m} \\ w_{r2,m}-w_{i2,m} \\ w_{r2,m}+w_{i2,m} \end{bmatrix}$$

Here the elements of the matrix product

$$(4.21)\qquad A_m = D_m E^{-1}{}_m \quad \textit{may be computed as follows}:$$

$$(A_m)_{11} = \frac{-b_{2,m}F_{1,m}\cos\theta_m + b_{1,m}F_{2,m}\cos\phi_m}{\left(F_{1,m}b_{2,m}-F_{2,m}b_{1,m}\right)},$$

$$\left(A_m\right)_{12} = \frac{l\left[-a_{2,m}F_{1,m}\sin\theta_m + a_{1,m}F_{2,m}\sin\phi_m\right]}{\left(a_{2,m}-a_{1,m}\right)},$$

$$\left(A_m\right)_{13} = \frac{l\left[F_{1,m}\sin\theta_m - F_{2,m}\sin\phi_m\right]}{\left(a_{2,m}-a_{1,m}\right)},$$

$$\left(A_m\right)_{14} = \frac{F_{1,m}F_{1,m}\left[\cos\theta_m - \cos\phi_m\right]}{\left(F_{1,m}b_{2,m}-F_{2,m}b_{1,m}\right)},$$

$$\left(A_m\right)_{21} = \frac{l\left[-b_{2,m}\sin\theta_m + b_{1,m}\sin\phi_m\right]}{\left(F_{1,m}b_{2,m}-F_{2,m}b_{1,m}\right)},$$

$$(A_m)_{22} = \frac{\left[-a_{2,m}\cos\theta_m + a_{1,m}\cos\phi_m\right]}{\left(a_{2,m} - a_{1,m}\right)},$$

$$(A_m)_{23} = \frac{\left[\cos\theta_m - \cos\phi_m\right]}{\left(a_{2,m} - a_{1,m}\right)},$$

$$(A_m)_{24} = \frac{l\left[F_{1,m}\sin\theta_m - F_{2,m}\sin\phi_m\right]}{\left(F_{1,m}b_{2,m} - F_{2,m}b_{1,m}\right)},$$

$$(A_m)_{31} = \frac{l\left[a_{1,m}b_{2,m}\sin\theta_m + a_{2,m}b_{1,m}\sin\phi_m\right]}{\left(F_{1,m}b_{2,m} - F_{2,m}b_{1,m}\right)},$$

$$(A_m)_{32} = \frac{a_{1,m}a_{2,m}\left[-\cos\theta_m + \cos\phi_m\right]}{\left(a_{2,m} - a_{1,m}\right)},$$

$$(A_m)_{33} = \frac{a_{1,m}\cos\theta_m - a_{2,m}\cos\phi_{\partial m}}{\left(a_{2,m} - a_{1,m}\right)},$$

$$(A_m)_{34} = \frac{l\left[a_{1,m}F_{2,m}\sin\theta_m - a_{2,m}F_{1,m}\sin\phi_m\right]}{\left(F_{1,m}b_{2,m} - F_{2,m}b_{1,m}\right)},$$

$$(A_m)_{41} = \frac{b_{1,m}b_{2,m}\left[-\cos\theta_m + \cos\phi_m\right]}{\left(F_{1,m}b_{2,m} - F_{2,m}b_{1,m}\right)},$$

$$(A_m)_{42} = \frac{l\left[-a_{2,m}b_{1,m}\sin\theta_m + a_{1,m}b_{2,m}\sin\phi_m\right]}{\left(a_{2,m} - a_{1,m}\right)},$$

$$(A_m)_{43} = \frac{l\left[b_{1,m}\sin\theta_m - b_{2,m}\sin\phi_m\right]}{\left(a_{2,m} - a_{1,m}\right)},$$

$$(A_m)_{44} = \frac{\left[b_{1,m}F_{2,m}\cos\theta_m - B_{2,m}F\cos\phi_m\right]}{\left(F_{1,m}b_{2,m} - F_{2,m}b_{1,m}\right)}.$$

The boundary conditions require that the values of displacements and incremental stresses at the top of the m$^{th}$ layer be the same as the values computed at the bottom of the (m-1)$^{th}$ layer .We may write

$$(4.22)\quad \begin{bmatrix} u_m \\ w_m \\ (\Delta f_x)_m \\ (\Delta f_z)_m \end{bmatrix} = A_m A_{m-1} \begin{bmatrix} u_{m-2} \\ w_{m-2} \\ (\Delta f_x)_{m-2} \\ (\Delta f_x)_{m-2} \end{bmatrix}$$

By repeated application of equation (45), we have,

$$(4.23)\qquad \begin{bmatrix} u_{n-1} \\ w_{n-1} \\ (\Delta f_x)_{n-1} \\ (\Delta f_z)_{n-1} \end{bmatrix} = A_{n-1}A_{n-2}......A_1 \begin{bmatrix} u_0 \\ w_0 \\ (\Delta f_x)_0 \\ (\Delta f_z)_0 \end{bmatrix},$$

where $u_0 = u_1(z=0),\ w_0 = w_1(z=0)$ are displacements and $(\Delta f_x)_0 = (\Delta f_x)_1\,(z=0), (\Delta f_z)_0 = (\Delta f_x)_1\,(z=0)$ are incremental stresses at the free surface: From equations (40) and (45), we obtain

$$(4.24)\qquad \begin{bmatrix} W_{r1,n} + W_{i1,n} \\ W_{r1,n} - W_{i1,n} \\ W_{r2,n} - W_{i2,n} \\ W_{r2,n} + W_{i2,n} \end{bmatrix} = E^{-1}{}_n\ A_{n-1}A_{n-2}......A_1 \begin{bmatrix} u_0 \\ w_0 \\ (\Delta f_x)_0 \\ (\Delta f_x)_0 \end{bmatrix},$$

(4.25) Putting $J_n = A_{n-1}A_{n-2}......A_{1,}\ J_n$

in equation (4.24), we get

$$(4.26)\qquad \begin{bmatrix} W_{r1,n} + W_{i1,n} \\ W_{r1,n} - W_{i1,n} \\ W_{r2,n} - W_{i2,n} \\ W_{r2,n} + W_{i2,n} \end{bmatrix} = J_n \begin{bmatrix} u_0 \\ w_0 \\ (\Delta f_x)_0 \\ (\Delta f_x)_0 \end{bmatrix}$$

In general, equation (4.26) is equally applicable to surface waves or to waves transmitted through the initially stressed layered medium.

## 5. Rayleigh Waves

Here we restrict our analysis to surface waves, in which there are no stresses across the free surface, so that $(\Delta f_x)_0 = (\Delta f_z)_0 = 0$, and there are no sources at infinity, so that $W_{i1,n} = W_{i2,n} = 0$. Then equation (4.26) reduces to

(5.1)
$$\begin{bmatrix} W_{r1,n} \\ W_{r1,n} \\ W_{r2,n} \\ W_{r2,n} \end{bmatrix} = J_n \begin{bmatrix} u_0 \\ w_0 \\ 0 \\ 0 \end{bmatrix},$$

Equation (5.1) gives

(5.2)
$$\begin{aligned} W_{r1,n} &= (J_n)_{11} u_0 + (J_n)_{12} w_0, \\ W_{r1,n} &= (J_n)_{21} u_0 + (J_n)_{22} w_0, \\ W_{r2,n} &= (J_n)_{31} u_0 + (J_n)_{32} w_0, \\ W_{r2,n} &= (J_n)_{41} u_0 + (J_n)_{42} w_0. \end{aligned}$$

On simplification equation (5.2), gives

(5.3)
$$\frac{u_0}{w_0} = \frac{(J_n)_{22} - (J_n)_{12}}{(J_n)_{11} - (J_n)_{21}} = \frac{(J_n)_{42} - (J_n)_{32}}{(J_n)_{31} - (J_n)_{41}}.$$

## 6. Special Cases

### 6.1 Single layer case

We take n=2, i.e. a single layer over a half-space. For this, equation (4.25) yields

(6.1) $J_2 = E^{-1}{}_2 A_1$.

Using equations (4.15), (4.19) and (4.21) in equation (6.1), we have

$$(J_2)_{11} = \frac{l\left[\left(a_{2,2}b_{2,1} - a_{1,1}b_{2,1}\right)\sin\theta_1 + \left(a_{2,1}b_{1,1} - a_{2,2}b_{1,1}\right)\sin\phi_1\right]}{\left(F_{1,1}b_{2,1} - F_{2,1}b_{1,1}\right)\left(a_{2,2} - a_{2,1}\right)},$$

$$(J_2)_{12} = \frac{a_{2,1}\left[\left(a_{2,2} - a_{1,1}\right)\cos\theta_1 + a_{1,1}\left(a_{2,1} - a_{2,2}\right)\cos\phi_1\right]}{\left(a_{2,1} - a_{1,1}\right)\left(a_{2,2} - a_{1,2}\right)},$$

$$(J_2)_{21} = \frac{b_{2,1}\left[\left(F_{2,2}b_{1,1} - F_{1,1}b_{2,2}\right)\cos\theta_1 + b_{1,1}\left(F_{2,1}b_{2,2} - F_{2,2}b_{2,1}\right)\cos\phi_1\right]}{\left(F_{1,1}b_{2,2} - F_{221}b_{1,2}\right)\left(F_{1,1}b_{2,1} - F_{2,1}b_{1,1}\right)},$$

(6.2)
$$(J_2)_{22} = \frac{l\left[a_{2,1}\left(F_{2,2}b_{1,1} - F_{1,1}b_{2,2}\right)\sin\theta_1 + a_{1,1}\left(F_{2,1}b_{2,2} - F_{2,2}b_{2,1}\right)\sin\phi_1\right]}{\left(F_{1,2}b_{2,2} - F_{2,1}b_{1,2}\right)\left(a_{2,1} - a_{1,1}\right)},$$

$$(J_2)_{31} = \frac{b_{2,1}\left[\left(F_{1,1}b_{2,2} - F_{1,2}b_{1,1}\right)\cos\theta_1 + b_{1,1}\left(F_{1,2}b_{2,1} - F_{2,1}b_{1,2}\right)\cos\phi_1\right]}{\left(F_{1,1}b_{2,1} - F_{221}b_{1,1}\right)\left(F_{1,1}b_{2,2} - F_{2,2}b_{1,2}\right)},$$

$$(J_2)_{32} = \frac{l\left[a_{2,1}\left(F_{1,1}b_{1,2} - F_{1,2}b_{1,1}\right)\sin\theta_1 + a_{1,1}\left(F_{1,2}b_{1,2} - F_{2,1}b_{1,2}\right)\sin\phi_1\right]}{\left(F_{1,2}b_{2,2} - F_{2,2}b_{1,2}\right)\left(a_{2,1} - a_{1,1}\right)},$$

$$(J_2)_{41} = \frac{l\left[b_{2,1}\left(a_{1,1} - a_{1,2}\right)\sin\theta_1 + b_{1,1}\left(a_{1,2} - a_{2,1}\right)\sin\phi_1\right]}{\left(F_{1,1}b_{2,1} - F_{2,1}b_{1,1}\right)\left(a_{2,2} - a_{1,2}\right)},$$

$$(J_2)_{42} = \frac{a_{2,1}\left[\left(a_{1,1} - a_{1,2}\right)\cos\theta_1 + a_{1,1}\left(a_{1,2} - a_{2,1}\right)\cos\phi_1\right]}{\left(a_{2,1} - a_{1,1}\right)\left(a_{2,2} - a_{1,2}\right)}$$

### 6.2. Two layer case

We take n=3, that is two layers lying over a half-space.Then using equations and following the same procedure of simplification as for single layer, case, we have from equation(4.25)

$$(J_3)_{22} = \frac{\left(B_{12}b_{2,3} - F_{2,3}B_{41}\right)}{\left(F_{1,3}b_{2,3} - F_{2,3}b_{1,3}\right)},$$

$$(J_3)_{31} = \frac{\left(-B_{11}b_{1,3} - F_{1,3}B_{41}\right)}{\left(F_{1,3}b_{2,3} - F_{2,3}b_{1,3}\right)},$$

$$(J_3)_{32} = \frac{\left(-B_{12}b_{1,3} + F_{1,3}B_{42}\right)}{\left(F_{1,3}b_{2,3} - F_{2,3}b_{1,3}\right)},$$

$$(J_3)_{41} = \frac{\left(B_{21}a_{1,3} - B_{31}\right)}{\left(a_{2,3} - a_{1,3}\right)},$$

$$(J_3)_{42} = \frac{\left(B_{22}a_{1,3} - B_{32}\right)}{\left(a_{2,3} - a_{1,3}\right)},$$

here

$$B_{11} = \left(A_2\right)_{11}\left(A_2\right)_{11} + \left(A_2\right)_{12}\left(A_1\right)_{21} + \left(A_2\right)_{13}\left(A_1\right)_{31} + \left(A_2\right)_{14}\left(A_1\right)_{41},$$

$$B_{12} = \left(A_2\right)_{11}\left(A_1\right)_{12} + \left(A_2\right)_{12}\left(A_1\right)_{22} + \left(A_2\right)_{13}\left(A_1\right)_{32} + \left(A_2\right)_{14}\left(A_1\right)_{42},$$

$$B_{21} = \left(A_2\right)_{21}\left(A_1\right)_{11} + \left(A_2\right)_{22}\left(A_1\right)_{21} + \left(A_2\right)_{23}\left(A_1\right)_{31} + \left(A_2\right)_{24}\left(A_1\right)_{41},$$

$B_{22} = (A_2)_{21}(A_2)_{12} + (A_2)_{12}(A_1)_{22} + (A_2)_{23}(A_1)_{32} + (A_2)_{24}(A_1)_{42}$,

(6.6) $B_{31} = (A_2)_{31}(A_1)_{11} + (A_2)_{32}(A_1)_{21} + (A_2)_{33}(A_1)_{31} + (A_2)_{34}(A_1)_{41}$,

$B_{32} = (A_2)_{31}(A_1)_{12} + (A_2)_{32}(A_1)_{22} + (A_2)_{33}(A_1)_{32} + (A_2)_{34}(A_1)_{42}$,

$B_{41} = (A_2)_{41}(A_1)_{11} + (A_2)_{42}(A_1)_{21} + (A_2)_{43}(A_1)_{31} + (A_2)_{44}(A_1)_{41}$,

$B_{42} = (A_2)_{41}(A_1)_{12} + (A_2)_{42}(A_1)_{22} + (A_2)_{43}(A_1)_{32} + (A_2)_{44}(A_1)_{41}$

*and*

(6.7) $(A_1)_{11}, (A_1)_{12}, (A_1)_{21}, (A_1)_{31}, (A_1)_{32}, (A_1)_{41}, (A_1)_{42,} (A_2)_{11}, (A_2)_{12}, (A_2)_{21}$, $(A_2)_{31}, (A_2)_{32}, (A_2)_{41}$ *and* $(A_2)_{42}$

are obtained from equation (4.21).

## 7. Conclusions

We have derived the dispersion equations for Rayleigh waves in prestressed multilayered elastic half-space. Particular cases of Rayleigh waves in layered half-space and two layered half-space have been considered in detail.

**Acknowledgement**: One of the author (I.S.G) is thankful to UGC for providing financial assistance through MRP.

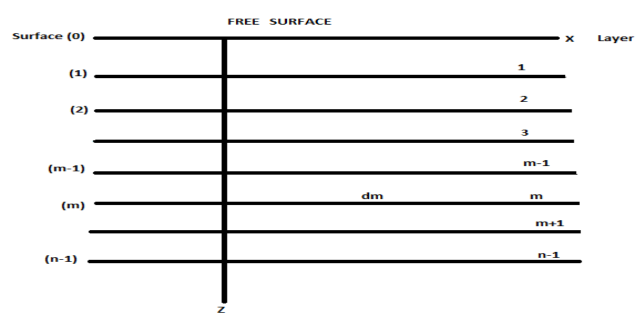
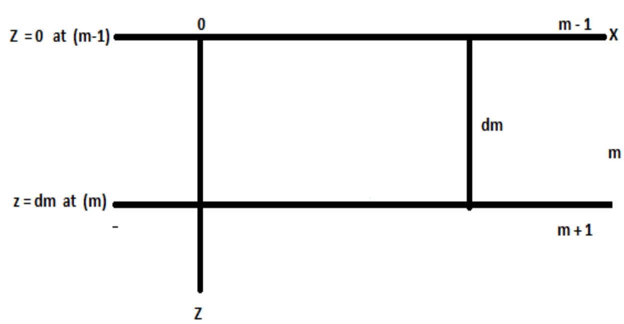



Fig.1 Geometry of the problem



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