ISSN 0974 - 9373

Vol. 16 No.4 (2012) **Journal of International Academy of Physical Sciences** pp. 329-341

Matsumoto Change of Finsler Metric

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(Received October 10, 2012)

Abstract: The purpose of the present paper is to find the necessary and sufficient conditions under which a Matsumoto change becomes a projective change. We have also found the conditions under which a Matsumoto change of Douglas space becomes a Douglas space. The Matsumoto change of a Riemannian space has been discussed as a particular case.
Keywords: Matsumoto change, Projective change, Finsler space, Douglas space.
2000 Mathematics Subject Classification: 53B40, 53C60

1. Preliminaries

Let $F^n = (M^n, L)$ be a Finsler space equipped with the fundamental function *L(x, y)* on the smooth manifold M^n. Let $\beta = b_i(x)y^i$ be a one-form on the manifold M^n, then $L \to \dfrac{L^2}{L-\beta}$ is called Matsumoto change of Finsler metric. If we write $\bar{L} = \dfrac{L^2}{L-\beta}$ and $\bar{F}^n = (M^n, \bar{L})$, then the Finsler space $\bar{F}^n$ is said to be obtained from F^n by a Matsumoto change. The quantities corresponding to $\bar{F}^n$ are denoted by putting bar over those quantities.

The fundamental metric tensor g_{ij}, the normalized element of support l_i and angular metric tensor h_{ij} of F^n are given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}, \qquad l_i = \frac{\partial L}{\partial y^i} \qquad \text{and} \qquad h_{ij} = L\frac{\partial^2 L}{\partial y^i \partial y^j} = g_{ij} - l_i l_j .$$

We shall denote the partial derivative with respect to x^i and y^i by ∂_i and $\dot{\partial}_i$ respectively and write

$$L_i = \dot{\partial}_i L, \qquad L_{ij} = \dot{\partial}_i \dot{\partial}_j L, \qquad L_{ijk} = \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L.$$

Then

$$L_i = l_i, \qquad L^{-1} h_{ij} = L_{ij}.$$

The geodesics of F^n are given by the system of differential equations

$$\frac{d^2 x^i}{ds^2} + 2G^i\left(x, \frac{dx}{ds}\right) = 0,$$

where $G^i(x, y)$ are positively homogeneous of degree two in y^i and are given by

$$\text{(1.1)} \qquad 2G^i = g^{ij}(y^r \dot{\partial}_j \partial_r F - \partial_j F), \qquad F = \frac{L^2}{2},$$

where g^{ij} are the inverse of g_{ij}.

The well known Berwald connection $B\Gamma = (G^i_{jk}, G^i_j)$ of a Finsler space is constructed from the quantity G^i appearing in the equation of geodesic and is given by[1]

$$G^i_j = \dot{\partial}_j G^i, \qquad G^i_{jk} = \dot{\partial}_k G^i_j.$$

The Cartan's connection $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$ is constructed from the metric function L by the following five axioms[1]:

$$\text{(i)} \quad g_{ij|k} = 0 \qquad \text{(ii)} \quad g_{ij}|_k = 0 \qquad \text{(iii)} \quad F^i_{jk} = F^i_{kj}$$
$$\text{(iv)} \quad F^i_{0k} = G^i_k, \qquad \text{(v)} \quad C^i_{jk} = C^i_{kj},$$

where $_{|k}$ and $|_k$ denote h- and v-covariant derivatives with respect to $C\Gamma$. It is clear that the h-covariant derivative of L with respect to $B\Gamma$ and $C\Gamma$ is the same and vanishes identically. Furthermore, the h-covariant derivatives of L_i, L_{ij} with respect to $C\Gamma$ are also zero.

We shall write

$$\text{(1.2)} \qquad 2r_{ij} = b_{i|j} + b_{j|i}, \qquad 2s_{ij} = b_{i|j} - b_{j|i}.$$

2. Matsumoto Change of Finsler metric

The Matsumoto change of Finsler metric L is given by

$$(2.1) \qquad \overline{L} = \frac{L^2}{L-\beta}, \qquad \text{where} \qquad \beta(x,y) = b_i(x)\, y^i.$$

We may put

$$(2.2) \qquad \overline{G}^i = G^i + D^i.$$

Then $\overline{G}^i_j = G^i_j + D^i_j$ and $\overline{G}^i_{jk} = G^i_{jk} + D^i_{jk}$, where $D^i_j = \dot\partial_j D^i$ and $D^i_{jk} = \dot\partial_k D^i_j$. The tensors D^i, D^i_j and D^i_{jk} are positively homogeneous in y^i of degree two, one and zero respectively.

To find D^i we deal with equation[2] $L_{ij|k} = 0$, i.e.

$$(2.3) \qquad \partial_k L_{ij} - L_{ijr} G^r_k - L_{rj} F^r_{ik} - L_{ir} F^r_{jk} = 0.$$

Since $\dot\partial_i \beta = b_i$, from (2.1), we have

$$(2.4) \quad \text{(a)} \quad \overline{L}_i = \frac{L^2 - 2\beta L}{(L-\beta)^2} L_i + \frac{L^2}{(L-\beta)^2} b_i$$

$$\text{(b)} \quad \overline{L}_{ij} = \frac{L^2 - 2\beta L}{(L-\beta)^2} L_{ij} + \frac{2}{(L-\beta)^3}[\beta^2 L_i L_j - \beta L(L_i b_j + L_j b_i) + L^2 b_i b_j],$$

$$\text{(c)} \quad \begin{aligned} \partial_j \overline{L}_i &= \frac{L^2 - 2\beta L}{(L-\beta)^2} \partial_j L_i + \frac{2\beta}{(L-\beta)^3} (\beta L_i - L b_i) \partial_j L \\ &\quad + \frac{2L}{(L-\beta)^3} (L b_i - \beta L_i) \partial_j \beta + \frac{L^2}{(L-\beta)^2} \partial_j b_i \end{aligned},$$

$$\text{(d)} \quad \begin{aligned} \partial_k \overline{L}_{ij} &= \frac{L^2 - 2\beta L}{(L-\beta)^2} \partial_k L_{ij} - \frac{2}{(L-\beta)^4} \{-\beta^2 (L-\beta) L_{ij} + 3\beta^2 L_i L_j \\ &\quad + (L^2 + 2\beta L) b_i b_j - (\beta^2 + 2\beta L)(L_i b_j + L_j b_i)\} \partial_k L \\ &\quad + \frac{2}{(L-\beta)^4} \{-\beta L (L-\beta) L_{ij} + (\beta^2 + 2\beta L) L_i L_j + 3L^2 b_i b_j \\ &\quad - (L^2 + 2\beta L)(L_i b_j + L_j b_i)\} \partial_k \beta + \frac{2\beta}{(L-\beta)^3} \end{aligned}$$

$$(\beta L_i - Lb_i)\partial_k L_j + \frac{2\beta}{(L-\beta)^3}(\beta L_j - Lb_j)\partial_k L_i$$
$$+\frac{2L}{(L-\beta)^3}(Lb_i - \beta L_i)\partial_k b_j + \frac{2L}{(L-\beta)^3}(Lb_j - \beta L_j)\partial_k b_i,$$

and

$$\text{(e)}\quad \begin{aligned}\overline{L}_{ijk} &= \frac{L^2 - 2\beta L}{(L-\beta)^2} L_{ijk} + \frac{2}{(L-\beta)^3}\{\beta^2(L_{ij}L_k + L_{jk}L_i + L_{ik}L_j) \\ &\quad - \beta L(L_{ij}b_k + L_{jk}b_i + L_{ik}b_j)\} \\ &\quad + \frac{2}{(L-\beta)^4}\{(\beta^2 + 2\beta L)(L_iL_jb_k + L_jL_kb_i + L_iL_kb_j) \\ &\quad - (L^2 + 2\beta L)(L_ib_jb_k + L_jb_ib_k + L_kb_ib_j) \\ &\quad - 3\beta^2 L_iL_jL_k + 3L^2 b_ib_jb_k\}.\end{aligned}$$

Since $\overline{L}_{ij|k} = 0$ in $\overline{F}^n$, after using (2.2), we have

$$\partial_k \overline{L}_{ij} - \overline{L}_{ijr}(G^r_k + D^r_k) - \overline{L}_{rj}(F^r_{ik} + {}^cD^r_{ik}) - \overline{L}_{ir}(F^r_{jk} + {}^cD^r_{jk}) = 0,$$

where $\overline{F}^i_{jk} - F^i_{jk} = {}^cD^i_{jk}$.

Substituting in the above equation the values of $\partial_k \overline{L}_{ij}$, $\overline{L}_{ir}$ and $\overline{L}_{ijr}$ from (2.4) and using (2.3) and then contracting the equation thus obtained with y^k, we get

$$\text{(2.5)}\quad \begin{aligned} 2\Bigg[&-\frac{L^2-2\beta L}{(L-\beta)^2}L_{ijr} - \frac{2\beta}{(L-\beta)^3}\{\beta(L_{ij}L_r + L_{jr}L_i + L_{ir}L_j) - L(L_{ij}b_r + L_{jr}b_i \\ &+ L_{ir}b_j)\} - \frac{2(\beta^2+2\beta L)}{(L-\beta)^4}(L_iL_jb_r + L_jL_rb_i + L_iL_rb_j) + \frac{2(L^2+2\beta L)}{(L-\beta)^4}(L_ib_jb_r \\ &+ L_jb_ib_r + L_rb_ib_j) + \frac{6\beta^2}{(L-\beta)^4}L_iL_jL_r - \frac{6L^2}{(L-\beta)^4}b_ib_jb_r\Bigg]D^r \\ &- \left[\frac{L^2-2\beta L}{(L-\beta)^2}L_{rj} + \frac{2}{(L-\beta)^3}\{\beta^2 L_rL_j - \beta L(L_rb_j + L_jb_r) + L^2 b_rb_j\}\right]D^r_i \\ &- \left[\frac{L^2-2\beta L}{(L-\beta)^2}L_{ir} + \frac{2}{(L-\beta)^3}\{\beta^2 L_iL_r - \beta L(L_ib_r + L_rb_i) + L^2 b_ib_r\}\right]D^r_j \end{aligned}$$

$$+\frac{2}{(L-\beta)^4}[-\beta L(L-\beta)L_{ij}+(\beta^2+2\beta L)L_iL_j+3L^2b_ib_j-(L^2+2\beta L)$$
$$(L_ib_j+L_jb_i)]r_{00}+\frac{2L}{(L-\beta)^3}(Lb_i-\beta L_i)(r_{j0}+s_{j0})$$
$$+\frac{2L}{(L-\beta)^3}(Lb_j-\beta L_j)(r_{i0}+s_{i0})=0.$$

where '0' stands for contraction with y^k viz. $r_{j0}=r_{jk}y^k$, $r_{00}=r_{ij}y^iy^j$ and we have used the fact that $D^i_{jk}y^k={}^cD^i_{jk}y^k=D^i_j$.[3]

Next, we deal with $\overline{L}_{i|j}=0$, that is $\partial_j\overline{L}_i-\overline{L}_{ir}\overline{G}^r_j-\overline{L}_r\overline{F}^r_{ij}=0$. Then

(2.6) $$\partial_j\overline{L}_i-\overline{L}_{ir}(G^r_j+D^r_j)-\overline{L}_r(F^r_{ij}+{}^cD^r_{ij})=0.$$

Putting the values of $\partial_j\overline{L}_i$, $\overline{L}_{ir}$ and $\overline{L}_r$ from (2.4) in (2.6) and using equation

$$L_{i|j}=\partial_jL_i-L_{ir}G^r_j-L_rF^r_{ij}=0,$$

and rearranging the terms, we get

$$\frac{L^2}{(L-\beta)^2}b_{i|j}=\left[\frac{L^2-2\beta L}{(L-\beta)^2}L_{ir}+\frac{2}{(L-\beta)^3}\{\beta^2L_iL_r-\beta L(L_ib_r+L_rb_i)+L^2b_ib_r\}\right]D^r_j$$
$$+\left[\frac{L^2-2\beta L}{(L-\beta)^2}L_r+\frac{L^2}{(L-\beta)^2}b_r\right]{}^cD^r_{ij}-\frac{2L}{(L-\beta)^3}(Lb_i-\beta L_i)(r_{j0}+s_{j0}),$$

which after using (1.2), gives

(2.7) $$\frac{2L^2}{(L-\beta)^2}r_{ij}=\left[\frac{L^2-2\beta L}{(L-\beta)^2}L_{ir}+\frac{2}{(L-\beta)^3}\{\beta^2L_iL_r-\beta L(L_ib_r+L_rb_i)+L^2b_ib_r\}\right]D^r_j$$
$$+\left[\frac{L^2-2\beta L}{(L-\beta)^2}L_{jr}+\frac{2}{(L-\beta)^3}\{\beta^2L_jL_r-\beta L(L_jb_r+L_rb_j)+L^2b_jb_r\}\right]D^r_i$$
$$-\frac{2L}{(L-\beta)^3}(Lb_i-\beta L_i)(r_{j0}+s_{j0})-\frac{2L}{(L-\beta)^3}(Lb_j-\beta L_j)(r_{i0}+s_{i0})$$
$$+2\left[\frac{L^2-2\beta L}{(L-\beta)^2}L_r+\frac{L^2}{(L-\beta)^2}b_r\right]{}^cD^r_{ij}$$

and

$$(2.8)\quad \frac{2L^2}{(L-\beta)^2}s_{ij}=\left[\frac{L^2-2\beta L}{(L-\beta)^2}L_{ir}+\frac{2}{(L-\beta)^3}\{\beta^2 L_i L_r-\beta L(L_i b_r+L_r b_i)+L^2 b_i b_r\}\right]D^r_j$$
$$-\frac{2L}{(L-\beta)^3}(Lb_i-\beta L_i)(r_{j0}+s_{j0})-\left[\frac{L^2-2\beta L}{(L-\beta)^2}L_{jr}+\frac{2}{(L-\beta)^3}\{\beta^2 L_j L_r\right.$$
$$\left.-\beta L(L_j b_r+L_r b_j)+L^2 b_j b_r\}\right]D^r_i+\frac{2L}{(L-\beta)^3}(Lb_j-\beta L_j)(r_{i0}+s_{i0}).$$

Subtracting (2.7) from (2.5) and contracting the resulting equation with y^i, we obtain

$$(2.9)\quad \left[-\frac{L^2-2\beta L}{(L-\beta)^2}L_{jr}-\frac{2}{(L-\beta)^3}\{\beta^2 L_j L_r-\beta L(L_j b_r+L_r b_j)+L^2 b_j b_r\}\right]D^r$$
$$-\frac{L}{(L-\beta)^3}(\beta L_j-Lb_j)r_{00}+\frac{L^2}{(L-\beta)^2}r_{j0}=\left[\frac{L^2-2\beta L}{(L-\beta)^2}L_r+\frac{L^2}{(L-\beta)^2}b_r\right]D^r_j.$$

Contracting (2.9) with y^j, we get

$$(2.10)\quad 2(L-2\beta)\,L_r D^r+2L\,b_r D^r=Lr_{00}.$$

Subtracting (2.8) from (2.5) and contracting the resulting equation with y^j, we get

$$(2.11)\quad \left[\frac{L^2-2\beta L}{(L-\beta)^2}L_{ir}+\frac{2}{(L-\beta)^3}\{\beta^2 L_i L_r-\beta L(L_i b_r+L_r b_i)+L^2 b_i b_r\}\right]D^r$$
$$=\frac{L}{(L-\beta)^3}(Lb_i-\beta L_i)r_{00}+\frac{L^2}{(L-\beta)^2}s_{i0}.$$

In view of $LL_{ir}=g_{ir}-L_i L_r$, the equation (2.11) can be written as

$$(2.12)\quad \frac{L-2\beta}{(L-\beta)^2}g_{ir}D^r-\left\{\frac{(L^2-3\beta L)L_i+2\beta Lb_i}{(L-\beta)^3}\right\}L_r D^r$$
$$-\left\{\frac{2\beta LL_i-2L^2 b_i}{(L-\beta)^3}\right\}b_r D^r=\frac{L}{(L-\beta)^3}(Lb_i-\beta L_i)r_{00}+\frac{L^2}{(L-\beta)^2}s_{i0}.$$

Contracting (2.12) with $b^i = g^{ij} b_j$, we get

(2.13) $(3\beta^2 - \beta L - 2\beta L b^2) L_r D^r + (L^2 - 3\beta L + 2b^2 L^2) b_r D^r = L^2 (L - \beta) s_0 + (b^2 L^2 - \beta^2) r_{00}$,

where we have written s_0 for $s_{r0} b^r$.

The equations (2.10) and (2.13) constitute the system of algebraic equations in $L_r D^r$ and $b_r D^r$ whose solution is given by

$$(2.14) \qquad b_r D^r = \frac{2L^2 (L - 2\beta) s_0 + (2b^2 L^2 + \beta L - 4\beta^2) r_{00}}{2L\{(1 + 2b^2) L - 3\beta\}}$$

and

$$(2.15) \qquad L_r D^r = -\frac{2L^2 s_0 - (L - 2\beta) r_{00}}{2\{(1 + 2b^2) L - 3\beta\}}.$$

Contracting (2.12) by g^{ij} and putting the values of $b_r D^r$ and $L_r D^r$ from (2.14) and (2.15) respectively, we get

$$(2.16) \quad D^i = \frac{(L - 4\beta)\{(L - 2\beta) r_{00} - 2L^2 s_0\}}{2L(L - 2\beta)\{(1 + 2b^2) L - 3\beta\}} y^i + \frac{L\{(L - 2\beta) r_{00} - 2L^2 s_0\}}{(L - 2\beta)\{(1 + 2b^2) L - 3\beta\}} b^i + \frac{L^2}{L - 2\beta} s_0^i,$$

where $l^i = \dfrac{y^i}{L}$.

Proposition 2.1. *The difference tensor* $D^i = \overline{G}^i - G^i$ *of Matsumoto change of Finsler metric is given by (2.16).*

3. Projective Change of Finsler Metric

The Finsler space $\overline{F}^n$ is said to be projective to Finsler space F^n if every geodesic of F^n is transformed to a geodesic of $\overline{F}^n$. It is well known that the change $L \to \overline{L}$ is projective if $\overline{G}^i = G^i + P(x, y) y^i$, where $P(x, y)$ is a homogeneous scalar function of degree one in y^i, called projective factor[4].

Thus from (2.2) it follows that $L \to \overline{L}$ is projective iff $D^i = P y^i$. Now we consider that the Matsumoto change $L \to \overline{L} = \dfrac{L^2}{L - \beta}$ is projective. Then from equation (2.16), we have

(3.1) $$Py^i = \frac{(L-4\beta)\{(L-2\beta)r_{00}-2L^2s_0\}}{2L(L-2\beta)\{(1+2b^2)L-3\beta\}}y^i + \frac{L\{(L-2\beta)r_{00}-2L^2s_0\}}{(L-2\beta)\{(1+2b^2)L-3\beta\}}b^i + \frac{L^2}{L-2\beta}s_0^i.$$

Contracting (3.1) with $y_i(=g_{ij}y^j)$ and using the fact that $s_0^i y_i = 0$ and $y_i y^i = L^2$, we get

(3.2) $$P = \frac{(L-2\beta)r_{00}-2L^2s_0}{2L\{(1+2b^2)L-3\beta\}}.$$

Putting the value of P from (3.2) in (3.1), we get

(3.3) $$\beta\{(L-2\beta)r_{00}-2L^2s_0\}y^i = L^2\{(L-2\beta)r_{00}-2L^2s_0\}b^i + L^3\{(1+2b^2)L-3\beta\}s_0^i.$$

Transvecting (3.3) by b^i, we get

(3.4) $$r_{00} = (L-\beta)\frac{s_0}{\Delta}, \quad \text{where} \quad \Delta = \left(\frac{\beta}{L}\right)^2 - b^2 \neq 0.$$

Substituting the value of r_{00} from (3.4) in (3.2), we get

(3.5) $$P = \frac{s_0}{2\Delta}.$$

Eliminating P and r_{00} from (3.5), (3.4) and (3.1), we get

(3.6) $$s_0^i = \left[\frac{\beta}{L^2}y^i - b^i\right]\frac{s_0}{\Delta}.$$

The equations (3.4) and (3.6) give the necessary conditions under which a Matsumoto change becomes a projective change.

Conversely, if conditions (3.4) and (3.6) are satisfied, then putting these conditions in (2.16), we get

$$D^i = \frac{s_0}{2\Delta}y^i, \text{ i.e. } D^i = Py^i, \text{ where } \quad P = \frac{s_0}{2\Delta}.$$

Thus $\bar{F}^n$ is projective to F^n.

Theorem 3.1. *The Matsumoto change of a Finsler space is projective if and only if (3.4) and (3.6) hold.*

Let us assume that L is the metric of a Riemannian space, that is $L=\alpha=\sqrt{a_{ij}(x)y^i y^j}$. Then $\overline{L}=\dfrac{\alpha^2}{\alpha-\beta}$, which is the metric of Matsumoto space. In this case $b_{i|j}=b_{i;j}$ where $;j$ denotes the covariant derivative with respect to Christoffel symbols constructed from Riemannian metric α. Thus r_{ij} and s_{ij} are functions of co-ordinates only, and in view of theorem (3.1), it follows that the Riemannian space is projective to Matsumoto space iff $r_{00}=(\alpha-\beta)\dfrac{s_0}{\Delta}$ and $s_0^i=\left(\dfrac{\beta}{\alpha^2}y^i-b^i\right)\dfrac{s_0}{\Delta}$, where $\Delta=\left(\dfrac{\beta}{\alpha}\right)^2-b^2\neq 0$. These equations may be written as

(3.7) (a) $\quad r_{00}(\beta^2-b^2\alpha^2)=\alpha^2(\alpha-\beta)s_0$

(b) $\quad s_0^i(\beta^2-b^2\alpha^2)=(\beta y^i-\alpha^2 b^i)s_0$.

The equation (3.7)(b) can be written as

$$(s_j^i b_h b_k+s_h^i b_j b_k+s_k^i b_j b_h)-b^2(s_j^i a_{hk}+s_h^i a_{jk}+s_k^i a_{jh})=\frac{1}{2}[(b_h s_k+b_k s_h)\delta_j^i+ (b_j s_k+b_k s_j)\delta_h^i+(b_j s_h+b_h s_j)\delta_k^i]-b^i(a_{hk}s_j+a_{hj}s_k+a_{kj}s_h).$$

Contracting this equation with $i=j$, we get

(3.8) $\quad (s_h b_k+s_k b_h)=0,$ for $n>2$.

Transvecting (3.8) by b^h, we get $b^2 s_k=0$, which implies that $s_k=0$ provided $b^2\neq 0$. Therefore we have $s_0^i=0$, $s_0=0$ and (3.7)(a) gives $r_{00}=0$ as $\beta^2-b^2\alpha^2\neq 0$. Consequently $r_{ij}=0$, $s_{ij}=0$. Hence $b_{i;j}=0$, i.e. the pair (α,β) is parallel pair.

Conversely, if $b_{i;j}=0$, then equation (3.7)(a) and (b) hold identically. Thus we have

Theorem 3.2. *The Riemannian space with metric α is projective to Matsumoto space with metric $\dfrac{\alpha^2}{\alpha-\beta}$ iff the (α,β) is parallel pair.*

4. Matsumoto Change of Douglas Space

The Finsler space F^n is called a Douglas space iff $G^i y^j - G^j y^i$ is homogeneous polynomial of degree three in y^i.[5] We shall write *hp (r)* to denote a homogeneous polynomial in y^i of degree *r*. If we write $B^{ij} = D^i y^j - D^j y^i$, then from (2.16), we get

$$B^{ij} = \frac{L\{(L-2\beta)r_{00} - 2L^2 s_0\}}{(L-2\beta)\{(1+2b^2)L-3\beta\}}(b^i y^j - b^j y^i) + \frac{L^2}{L-2\beta}(s_0^i y^j - s_0^j y^i). \tag{4.1}$$

If a Douglas space is transformed to a Douglas space by a Matsumoto change (2.1), then B^{ij} must be *hp (3)* and vice-versa.

Theorem 4.1. *The Matsumoto change of Douglas space is a Douglas space iff* B^{ij} *given by (4.1) is hp (3).*

Since Riemannian space is a Douglas space, in the following we discuss the Matsumoto change of Riemannian space and find the condition under which the same is a Douglas space.

Let us assume that L is a Riemannian metric α, then $\bar{L} = \dfrac{\alpha^2}{\alpha - \beta}$ which is the metric of Matsumoto space. Therefore we find the condition for Finsler space $\bar{F}^n$ to be Douglas space by using theorem (4.1). In this case F^n is a Douglas space. Therefore $\bar{F}^n$ is a Douglas space iff B^{ij} is *hp(3)*. When L is a Riemannian metric, then r_{ij}, s_{ij}, s^i_j, s_j are functions of co-ordinates only and h-covariant derivative in F^n is nothing but covariant derivative with respect to Riemannian Christoffel symbol.
For $L = \alpha$, the equation (4.1) can be written as

$$\begin{aligned} &\{(1+2b^2)\alpha^2 + 6\beta^2\}B^{ij} + 3\alpha^2\beta(s_0^i y^j - s_0^j y^i) - \alpha^2 r_{00}(b^i y^j - b^j y^i) \\ &-\alpha[(5+4b^2)\beta B^{ij} + (1+2b^2)\alpha^2(s_0^i y^j - s_0^j y^i) - 2(\alpha^2 s_0 + r_{00}\beta)(b^i y^j - b^j y^i)] = 0. \end{aligned} \tag{4.2}$$

Since α is an irrational function in y^i, therefore equating to zero, the rational and irrational terms in y^i of equation (4.2), we get

$$\{(1+2b^2)\alpha^2 + 6\beta^2\}B^{ij} + 3\alpha^2\beta(s_0^i y^j - s_0^j y^i) - \alpha^2 r_{00}(b^i y^j - b^j y^i) = 0 \tag{4.3}$$

and

(4.4) $(5+4b^2)\beta B^{ij}+(1+2b^2)\alpha^2(s_0^i y^j - s_0^j y^i)-2(\alpha^2 s_0+\beta r_{00})(b^i y^j - b^j y^i)=0.$

Eliminating B^{ij} from equations (4.3) and (4.4), we get

(4.5) $$A(s_0^i y^j - s_0^j y^i)+B(b^i y^j - b^j y^i)=0,$$

where we put

$$A=\{9\beta^2-(1+2b^2)\alpha^2\},$$

$$B=[2s_0\{(1+2b^2)\alpha^2+6\beta^2\}-3\beta r_{00}]\alpha^2+12\beta^3 r_{00}.$$

Transvecting (4.5) by $b_i y_j$, we get

(4.6) $$A\alpha^2 s_0 + B(b^2\alpha^2-\beta^2)=0.$$

Since $-12r_{00}\beta^5$ is the only term of (4.6) which seemingly does not contain α^2, we must have $hp(5)$ u_5 such that

(4.7) $$r_{00}\beta^5=\alpha^2 u_5.$$

Then it will be better to divide our consideration into three cases as follows:

(i) $u_5=0$, (ii) $u_5\neq 0, \alpha^2\not\equiv 0 \pmod{\beta}$,
(iii) $u_5\neq 0$, $\alpha^2\equiv 0 \pmod{\beta}$.

The case (i) is simple: From (4.7) we have $r_{00}=0$ and (4.6) is reduced to

$$(\alpha^2-4\beta^2)\{(1+2b^2)\alpha^2-3\beta^2\}s_0=0,$$

which implies $s_0 = 0$ immediately.

Next we deal with the case (ii). The equation (4.7) shows the existence of a function $\lambda(x)$ satisfying $u_5=\lambda\beta^5$ and hence $r_{00}=\lambda\alpha^2$ then (4.6) is reduced to

(4.8) $$As_0 + (b^2\alpha^2 - \beta^2)[2s_0\{(1+2b^2)\alpha^2 + 6\beta^2\} - 3\lambda\beta(\alpha^2 - 4\beta^2)] = 0.$$

Since $-12(s_0 + \lambda\beta)\beta^4$ is the only term of (4.8) which seemingly does not contain α^2, hence we must have $hp(3)$ v_3 such that $(s_0 + \lambda\beta)\beta^4 = \alpha^2 v_3$. From $\alpha^2 \not\equiv 0$ (mod β) it follows that v_3 must vanish and hence $s_0 = -\lambda\beta$, i.e. $s_i = -\lambda b_i$. This on transvection by b^i, gives $\lambda b^2 = 0$. In case of $\lambda = 0$, we get $s_i = 0$ and $r_{00} = 0$. On the other hand, in case of $b^2 = 0$, equation (4.8) reduces to $\lambda\alpha^2\beta(\alpha^2 - 4\beta^2) = 0$, which implies $\lambda = 0$. Therefore both the cases (i) and (ii) lead to $s_0 = 0$ and $r_{00} = 0$. Hence (4.5) is reduced to $s_0^i y^j - s_0^j y^i = 0$, which on transvection by y_j gives $s_0^i = 0$. Finally $r_{ij} = s_{ij} = 0$ are concluded, that is $b_{i;j} = 0$.

Now we take the case (iii), wherein the following Lemma shall be used.

Lemma[6]. *If $\alpha^2 \equiv 0$ (mod β) i.e. $a_{ij}(x)y^i y^j$ contains $b_i(x)y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. In this case we have $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

Equation (4.7) is of the form $r_{00}\beta^4 = \delta u_5$, which must be reduced to $r_{00} = \delta v$, $v = v_i(x)y^i$. Consequently (4.6) is written as

(4.9) $$\delta(9\beta - \delta)s_0 - \beta\{2s_0(\delta + 6\beta) + 3v(4\beta - \delta)\} = 0.$$

Since $-\delta^2 s_0$ is the only term of (4.9) which seemingly does not contain β, there must exist a function $\lambda(x)$ satisfying $s_0 = \lambda\beta$, and the equation (4.9) is reduced to $3v = \lambda(\delta - 3\beta)$. Consequently we obtain

(4.10) $$r_{00} = \lambda\delta\left(\frac{\delta}{3} - \beta\right), \qquad s_0 = \lambda\beta.$$

Then (4.5) is written as $(s_0^i y^j - s_0^j y^i) + \lambda\delta(b^i y^j - b^j y^i) = 0$, which on transvection by y_j, leads to

(4.11) $$s_0^i = \lambda(y^i - \delta b^i).$$

Thus the equation (4.2) is written as

$$3(\alpha-2\beta)(\alpha-3\beta)B^{ij}=\lambda\delta^2(\beta\delta-5\alpha\beta+6\beta^2)(b^iy^j-b^jy^i).$$

From $\alpha^2=\beta\delta$ it follows that $(\alpha-2\beta)(\alpha-3\beta)=\beta\delta-5\alpha\beta+6\beta^2$, and hence

$$B^{ij}=\frac{\lambda\delta^2(b^iy^j-b^jy^i)}{3},$$

which are *hp(3)*. Equation (4.10) and (4.11) lead to

(4.12) $$b_{i;j}=\lambda\left(\frac{1}{3}d_i-b_i\right)d_j.$$

Thus, we get the following theorem which has been proved in[7]:

Theorem 4.2. *If $\alpha^2\not\equiv 0$ (mod β), then the Matsumoto space is Douglas space iff $b_{i;j}=0$.*

Theorem 4.3. *If $\alpha^2\equiv 0$ (mod β), then n = 2 and the Matsumoto space is a Douglas space iff $b_{i;j}$ is written in the form (4.12), where $\alpha^2=\beta\delta$, $\delta=d_i(x)y^i$ and $\lambda=\lambda(x)$.*

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