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# Absolute Banach Summability of Orthogonal Series

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**Abstract:** In this paper we have proved a theorem on generalized Nörlund summability of infinite series, which generalizes various known results. However, the theorem is as follows:

**Theorem:** *Let* $\{\Omega(n)\}$ *be positive sequence such that* $\left\{\frac{\Omega(n)}{n}\right\}$ *is a non-increasing sequence and the series* $\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$ *converges and* $\int_t^{\delta}\frac{\Phi_n(u)}{u}du = O\left(\frac{t}{\log\left(\frac{1}{t}\right)}\right)$ *as* $t\to 0$, $\delta$ *being some fixed positive constant then the orthogonal series* $\sum_{n=1}^{\infty}a_n\phi_n(x)$ *is summable* $|B|$ *at* $t = x$, *provided*

$$\sum_{n=1}^{\infty}k^2\log(n+k) = O(n\,\Omega(n)).$$

**Definitions and Notations:** Let $\{s_n\}$ be the sequence of partial sums of a series $\sum a_n$. Let the sequence $\{t_k(n)\}_{k=1}^{\infty}$ is defined by

(1.1) $$t_k(n) = \frac{1}{k}\sum_{v=0}^{k-1} s_{n+v} \quad , \qquad k \in N$$

If

(1.2) $\lim_{k\to\infty} t_k(n) = s$, a finite number, uniformly for all $n \in N$, then $\sum u_n$ is said to be Banach summable to $s$ [1].

Further if $\sum_{k=1}^{\infty} \left| t_k(n) - t_{k+1}(n) \right| < \infty$ uniformly for all $n \in N$,

then the series $\sum u_n$ is said to be absolute Banach summable or

simply $|B|$-summable.

**2.** Let $\{\phi_n\}$ be an orthogonal system defined in the interval $(a,b)$. We suppose that $f(x)$ belongs to $L^2(a,b)$ and

$$f(x) \approx \sum_{n=0}^{\infty} a_n \phi_n(x)$$

We denote by $E_n^{(2)}(f)$ the best approximation to $f(x)$ in the metric of $L^2$ by means of polynomials of $\phi_0(x), \phi_1(x), \ldots\ldots\ldots \phi_{n-1}(x)$.

It is well known that

$$E_n^{(2)}(f) = \left( \sum_{k=n}^{\infty} |a_k|^2 \right)^{1/2}$$

we write $\Delta\lambda_n = \lambda_n - \lambda_{n-1}$, for any sequence $\{\lambda_n\}$.

(2.1) $$g(k,t) = \frac{2}{\pi} \frac{1}{k(k+1)} \sum_{v=1}^{k} \frac{v}{(n+v)} (n+v)^{v-\beta} \frac{\Omega(t)}{t^2}$$

(2.2) $$J(k,u) = \frac{1}{F(1-\beta)} \int_u^\infty \frac{d}{dt} g(k,t)(t-u)^{-\beta}\, dt$$

$$\omega(k,u) = u^v J(k,u)$$

$[x]$ = greatest integer not exceeding x

$$U = \left(\frac{1}{u}\right) \text{ and } \tau = \left(\frac{1}{t}\right)$$

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## 1. Introduction

Ul'yanov[7] has proved the following theorems on $|C,\alpha|$ summability.

**Theorem A**: *If* $1 \geq \alpha > \frac{1}{2}$ *and* $\sum_{n=n_0}^{\infty} |a_n|^2 \log n (\log\log n)^{1+\epsilon}$ *converges, then the series* $\sum_{n=1}^{\infty} a_n \phi_n(x)$ *is summable* $|C,\alpha|$ *almost everywhere.*

**Theorem B:** *If* $0 < \alpha < \frac{1}{2}$ *and* $\sum_{n=n_0}^{\infty} |a_n|^2 n^{1-2\alpha} \log n (\log n)^{1+\epsilon}$ *converges, then the series* $\sum_{n=1}^{\infty} a_n \phi_n(x)$ *is summable* $|C,\alpha|$ *almost everywhere.*

**Theorem C:** *If* $1 \geq \alpha > \frac{1}{2}$ *and* $\sum_{n=n_0}^{\infty} n^{-1} (\log\log n)^{1+\epsilon} \left\{E_n^{(2)}(f)\right\}^2$ *converges, then the series* $\sum_{n=1}^{\infty} a_n \phi_n(x)$ *is summable* $|C,\alpha|$ *almost everywhere.*

**Theorem D:** *If* $0 < \alpha < \frac{1}{2}$ *and converges, then the series* $\sum_{n=1}^{\infty} a_n \phi_n(x)$ *is summable* $|C,\alpha|$ *almost everywhere.*

Generalizing the above theorems Okuyama[6] has proved following theorem for $|N, p_n|$ summability of orthogonal series.

**Theorem E:** *Let* $\{\Omega(n)\}$ *be positive sequence such that* ***Error! Reference source not found.*** *is a non-increasing sequence and the series* $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ *converges. Let* $\{p_n\}$ *be non-negative and non-increasing. If the series* $\sum_{n=1}^{\infty} |a_k|^2 \Omega(n)\omega_n$ *converges, then the orthogonal series* $\sum_{n=1}^{\infty} a_n\phi_n(x)$ *is summable* $|N, p_n|$ *almost everywhere.*

*where* $$\omega_k = \frac{1}{k}\sum_{n=1}^{\infty} \frac{n^2 p_n^2 p_{n-k}^2}{p_n}\left(\frac{P_n}{p_n} - \frac{P_{n-k}}{p_{n-k}}\right)^2$$ ***Error! Reference source not found.*** The main object of this paper is to generalize Theorem E for absolute Banach summability of orthogonal series.

**Main Theorem:** We establish our result in the form of following theorem.

**Theorem***: Let* $\{\Omega(n)\}$ *be positive sequence such that* $\left\{\frac{\Omega(n)}{n}\right\}$ *is a non-increasing sequence and the series* $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ *converges and*

$$\int_t^{\delta} \frac{\phi_n(u)}{u} du = O\left(\frac{t}{\log\left(\frac{1}{t}\right)}\right)$$ *as* $t \to 0$, $\delta$ *being some fixed positive constant*

*then the orthogonal series* $\sum_{n=1}^{\infty} a_n \phi_n(x)$ *is summable* $|B|$*at* $t = x$, *provided*

$\sum_{k=1}^{\infty} k^2 \log(n+k) = O(n\Omega(n))\cdot$

**Proof:** In order to prove the theorem. We have to prove that

$$\sum_{k=1}^{\infty} \left| t_k(n) - t_{k+1}(n) \right| = O(1)$$

Now taking

$$t_k(n) - t_{k+1}(n) = \frac{1}{k(k+1)} \sum_{v=1}^{k} v(n+v)\phi_{(n+v)}(t) n^{k-1} \text{ where } 0 < \gamma < 1$$

$$= \frac{1}{k(k+1)} \sum_{v=1}^{n} v(n+v)^{\gamma} \left(\frac{2}{\pi}\right) \int_0^{\pi} \frac{\Omega(t)}{t} \left\{ \frac{\phi_n(t)}{\log(n+t)} \right\} n^{k-1} dt$$

$$= t \left\{ \frac{1}{k(k+1)} \right\} \sum_{v=1}^{k} \frac{v}{(n+v)} (n+v)^{\gamma} \left(\frac{2}{\pi}\right) \int_0^{\pi} \frac{\Omega(t)}{t^2} \left\{ \frac{\phi_n(t)}{\log(n+t)} \right\} n^{k-1} dt$$

$$= \int_0^{\pi} \phi_n(t) \left[ \frac{2}{\pi} \frac{1}{k(k+1)} \sum_{v=1}^{k} v(n+v)^{\gamma-1} \frac{\Omega(t)}{t^2} \frac{n^{k-1}}{\log(n+t)} \right] dt$$

$$= \int_0^{\pi} \phi_n(t) \frac{\Omega(t)}{t^2} \frac{n^{k-1}}{\log(n+t)} \frac{d}{dt} g(k,t) dt$$

$$= \int_0^{\pi} \frac{d}{dt} g(k,t) \left\{ \int_0^{t} \left( \frac{\Omega(u)}{u^2} \frac{n^{k-1}}{\log(n+u)} du \right) \right\} dt$$

$$\leq \log t\left\{\sum_{k=2^m+1}^{2^{m+1}}\int_0^1 g(k,t)\frac{\Omega(t)}{t^2}\frac{n^{k-1}}{\log(n+t)}dt\right\}$$

$$\leq \log t\left[\sum_{k=2^m+1}^{2^{m+1}}\left\{\sum_{v=0}^{k} g(v,t)\frac{n^{k-1}}{\log(n+t)}+J(v,t)\frac{n^{k-1}}{\log(n+v)}\right\}\right]$$

$$\leq \log t\sum_{m=1}^{\infty}(2^m)\left[\sum_{k=2^m+1}^{2^{m+1}}\left\{\sum_{v=0}^{k}(n+v)\omega(k,v)\frac{n^{k-1}}{\log(n+v)}\right\}\right]$$

$$+\log t\sum_{m=1}^{\infty}(2^m)\left\{\sum_{k=2^m+1}^{2^{m+1}}\omega(k,v)\frac{n^{k-1}}{\log(n+k)}\right\}$$

$$=\sum\nolimits_1+\sum\nolimits_2$$

this estimation holds for any $k > -1$.

Now,

$$\sum\nolimits_1=\log t\sum_{n=1}^{\infty}(2^m)\left[\sum_{k=2^m+1}^{2^{m+1}}\left\{\sum_{v=0}^{k}(n+v)\omega(k,v)\frac{n^{k-1}}{\log(n+v)}\right\}\right]$$

$$=\sum_{k\leq\frac{1}{v}}O\big((n+v)\omega(k,v)\big)-O\left(\sum_{k<\frac{1}{v}}k^2\log(n+k)\right)$$

$$=O\left(n^v\right)O\left(n\Omega(n)\right)$$

$$=O(1)$$

Again $\sum_2=\log t\sum_{m=1}^{\infty}(2^m)\left\{\sum_{k=2^m+1}^{2^{m+1}}\omega(k,v)\frac{n^{k-1}}{\log(n+k)}\right\}$

$$= \log t \sum_{k>\frac{1}{v}} O\left\{\frac{2^m n^{(k-1)} n\Omega(n)}{k(k+1)\log(n+k)}\right\}$$

$$= \log t\, O(n^{v-1}) \sum_{k>\frac{1}{v}} \left\{\frac{2^m n\Omega(n)}{k(k+1)\log(n+k)}\right\}$$

$$= O(n^{v-1}) \sum_{k>\frac{1}{v}} \left\{\frac{2^m}{k(k+1)}\right\} O(n\Omega(n))$$

$$= O(n^{v-1}) O(n\Omega(n))$$

$$= O(1)$$

This completes the proof of the theorem.

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