ISSN 0974 - 9373

Vol. 17 No.3 (2013) **Journal of International Academy of Physical Sciences** pp. 277-285

A Study of P^*-Reducible Finsler Space with Douglas Tensor

Pooja Swaroop Saxena
Department of Mathematics
DIT University, Dehradun
Email: drpoojaswaroop@yahoo.com

Puneet Swaroop
Indian Institute of Remote Sensing, Dehradun

(Received June 13, 2013)

Abstract: In this paper we consider a Finsler space whose v (hv)-torsion tensor P_{ijk} is given by

$$P_{ijk} = \lambda C_{ijk} + \mu(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j)$$

where μ **&** λ are scalar functions positively homogeneous of degree one in y^i and call such a Finsler space as $P^* -$ reducible Finsler space. We have also proved that a $P^* -$ Randers space with vanishing Douglas tensor is a Riemannian space if the dimension is greater than three. Also we have worked out the role of $P^* -$reducibility condition in special Finsler spaces.

Key words: Finsler Space, C-reducible Finsler space, Douglas tensor.
AMS classification: 53B40

1. Introduction

A Finsler space F^n of dimension *n* is a differentiable manifold such that the length *s* of a curve *x'(t)* of *F''* is defined by the integral

$$s = \int L(x, \frac{dx}{dt})dt$$

The so-called fundamental function $L(x, y)(= L(x^i, y^i))$ is supposed to be differentiable for $y\neq(0)$ and to satisfy the usual regularity conditions in the variation calculus:

(i) positively homogeneous: $L(x, py) = pL(x, y), p > 0,$

(ii) positive: $L(x, y) > 0,\ for\ y \neq (0),$

(iii.) $g_{ij} = (\partial^2 L^2(x, y) / \partial y^i \partial y^j)/2$ is positive-definite,

(iv) $L(x,-y) = L(x, y).$

In the geometry of Finsler spaces, based on Cartan's connection, we have three kinds of covariant derivative of a tensor field[1,2]. These are denoted by $|_j$ & $\dot{\partial}_j$ and are called the h-covariant derivative and the v-covariant derivative respectively. There are three curvature tensors and three torsion tensors of Cartan's connection $C\Gamma$.

These are

(1) R_{hiik} h -curvature tensor,

(2) P_{hijk} hv -curvature tensor,

(3) S_{hijk} v -curvature tensor,

(4) $R_{ijk} = y^h R_{hijk}$ (v) h -torsion tensor

(5) $P_{ijk} = y^h P_{hijk}$ (v) hv -torsion tensor

(6) $C_{ijk} = \frac{1}{4}\dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L^2$ (h) hv - torsion tensor.

H. Izume[3, 4] defined a P^*-Finsler space which is characterized by the condition

(1.1) $P_{ijk} = \lambda C_{ijk},$

while M. Matsumoto[2,5] defined a C- reducible Finsler space which is characterized by the condition

(1.2) $$C_{ijk} = \frac{1}{n+1}\left(h_{ij} C_k + h_{jk} C_i + h_{ki} C_j\right)$$

where $h_{ij} = g_{ij} - l_i l_j$ is the angular metric tensor and $l_i = \dot{\partial}_i L$.

${}^*P-$ and P – reducible Finsler spaces are studied by P. N. Pandey[6,7].

The purpose of the present paper is to consider a Finsler space for which the tensor P_{ijk} assumes the form:

(1.3) $$P_{ijk} = \lambda C_{ijk} + \mu(h_{ij} C_k + h_{jk} C_i + h_{ki} C_j)$$

where λ and μ are scalar functions positively homogeneous of degree one in y^i. We shall call such a Finsler space as P^*-reducible Finsler space. Obviously this is a P_{ijk}- Finsler space if $\mu = 0$ while it is a C- reducible Finsler space if $\lambda = 0$.

2. Douglas Tensor

Let two Finsler spaces $F^n(M^n, L)$ and $\overline{F^n}(M^n, L)$ be defined on a common underlying manifold M^n. A diffeomorphism $F^n \to \overline{F^n}$ is called geodesic if it maps an arbitrary geodesic of F^n to a geodesic of $\overline{F^n}$. In this case the change $L \to \overline{L}$ of the metric is called projective. It is well-known that the mapping $F^n \to \overline{F^n}$ is geodesic iff there exists a scalar field p(x, y) satisfying the following equation

(2.1) $$\overline{G}^i = G^i + p(x, y) y^i, \quad p \neq 0.$$

The projective factor p(x, y) is a positive homogeneous function of degree one in y^i. From (2.1) we obtain the following equations

(2.2) $$\overline{G}^i_j = G^i_j + p\delta^i_j + p_j y^i, \quad p_j = \dot{\partial}_j p$$

(2.3) $$\overline{G}^i_{jk} = G^i_{jk} + p_j \delta^i_k + p_k \delta^i_j + p_{jk} y^i, \quad p_{jk} = \dot{\partial}_k p_j,$$

(2.4) $$\overline{G}^i_{jkl} = G^i_{jkl} + p_{jk}\delta^i_l + p_{lj}\delta^i_k + p_{kl}\delta^i_j + p_{jkl} y^i, \quad p_{jkl} = \dot{\partial}_l p_{jk}$$

Contracting the indices i and l in (2.4) and putting

$\overline{G}^r_{jkr} = \overline{G}_{jk}$ & $G^r{}_{jkr} = G_{jk}$, we get

(2.5) $p_{ij} = (\overline{G_{ij}} - G_{ij}) / (n+1)$

Directional difference of (2.5) gives

(2.6) $\mathrm{p}_{ijk} = (\overline{G}_{ij(k)} - G_{ij(k)}) / (n+1)$

where $\overline{G}_{ijk} = \dot{\partial}_k \overline{G}_{ij}$ & $G_{ijk} = \dot{\partial}_k G_{ij}$.

From (2.4), (2.5) and (2.6) we find that the tensor D^i_{jkh} defined by

(2.7) $D^i_{jkl} = G^i_{jkl} - (y^i G_{jk(l)} + \partial^i_j G_{kl} + \partial^i_k G_{jl} + \partial^i_l G_{jk}) / (n+1)$

which is invariant under geodesic mappings, that is

(2.8) $D^i_{jkl} = \overline{D}^i_{jkl}$

The tensor D^i_{jkh} is called Douglas tensor.

Now we give definitions of two special Finsler spaces.

Definition1. *An n-dimensional Finsler space (M^n, F) is called a Rander space*[8] *if*
L(x, y) = α(x, y) + β(x, y) where $\alpha(x, y) = \sqrt{a_{ij}(x) y^i y^j}$
is a Riemannian metric on M^n and β(x, y) = $b_i(x)y^i$ is a differentiable 1-form on M^n The metric L = α + β is called Randers metric.

Definition 2. *The Finsler metric L = α^2/β is called Kropina metric. The Finsler space $F^n = (M^n, L)$ such that L= α^2/β is called Kropina space.*

3. P^* – Randers space with Vanishing Douglas Tensor

Definition 5. *A Finsler space is said to be of Douglas type or Douglas space, iff the functions $G^i y^j - G^j y^i$ are homogeneous polynomials in (y^i) of degree three*[9].

Theorem 1. *A Finsler space is of Douglas type iff the Douglas tensor vanishes identically*[9].

Theorem 2. *For n > 3, in a C-reducible* P^{*}*–-Finsler space* $D^{i}_{jkl} = 0$ *holds*[4]. *If we consider a Randers change*

$$\overline{L}(x, y) \rightarrow \mathrm{L(x, y)} + \beta(\mathrm{x, y})$$

where β(x, y) is a closed one-form, then this change $\overline{L} \rightarrow L$ *s projective.*

Definition 6. *A Finsler space is called Landsberg space if the condition* $P_{ijk} = 0$ *holds.*

Theorem 3. *If there exists a Randers change with respect to a projective scalar* $p(x, y)$ *between a Landsberg and a* $\boldsymbol{P}^{*}$*-Finsler space (ful-filling the condition* $\overline{P}_{ijk} = p(x, y)\overline{C}_{ijk}$ *then* $p(x, y)$ *can be given by the equation*

(3.1) $$p(x, y) = e^{\varphi(x)}\overline{L}(x, y)$$

It is well-known that the Riemannian space is a special case of the Landsberg space. In a Riemannian space we have $D^{i}_{jkl} = 0$ and a $\boldsymbol{P}^{*}$-Randers space with a closed one-form $\beta(x, y)$ is a Finsler space with vanishing Douglas tensor.

Theorem 6. *A Randers space is a Douglas space iff* $\beta(x, y)$ *is a closed form*[9]. *Then*

(3.2) $$2G^{i} = \gamma^{i}_{jk} y^{j} y^{k} + \frac{r_{lm} y^{l} y^{m}}{\alpha + \beta} y^{i}$$

where $\gamma^{i}_{jk}(x)$ *is the Levi–Civita connection of a Riemannian space,* r_{lm} *is equal to* $b_{i;j}$.

Hence r_{lm} depends only on position. From Theorem 6. and (3.1) it follows that

$$\frac{r_{lm} y^l y^m}{\alpha+\beta} = e^{\phi(x)}(\alpha+\beta)$$

i.e.

$$\frac{r_{lm} y^l y^m}{\overline{L}} = e^{\phi(x)} \overline{L}$$

From the last equation we obtain

$$r_{lm} y^l y^m = e^{\phi(x)} \overline{L}^2$$

Differentiating twice this equation with respect to y^l and y^m we get

$$b_{i;j} = e^{\phi(x)} \overline{g}_{ij}$$

This means that the metrical tensor $\overline{g}_{ij}$ depends only on x, so we get the following

Theorem. *A P^* −Randers space with vanishing Douglas tensor is a Riemannian space if the dimension is greater than three.*

4. Two and Three Dimensional p*-Reducible Finsler Space

First of all we shall discuss the two dimensional Finsler space F^2. With reference to Berwald's frame (l_i, m_i) the angular metric tensor, (h) hv-torsion tensor and (v) hv-torsion tensors are given by[2]

(4.1) $$h_{ij} = m_i m_j, Lc_{ijk} = lm_i m_j m_k \quad P_{ijk} = I_{\backslash 0} m_i m_j m_k$$

where I is the main scalar. From above equations it follows that P_{ijk} may be written in the form (1.3) with $LI_{\backslash o} = I(\lambda+3\mu)$. Thus we have the following

Theorem (4.1): *Every two dimensional Finsler space is p*-reducible Finsler space, where the main scalar I satisfies*

$$LI_{\backslash o} = I(\lambda + 3\mu).$$

Next we deal with three-dimensional Finsler space. With respect to orthonormal frame $e^i_{(\alpha)}$, α = 0, 1, 2 , the (h) hv-torsion tensor is written as[10]

(4.2) $$Lc_{ijk} = c_{\alpha\beta\gamma} e_{(\alpha)i} e_{(\beta)j} e_{(\gamma)k}$$

where the scalar component $C_{\alpha\beta\gamma}$ are such that $C_{0\beta\gamma} = 0$.

$$c_{111} = H, c_{122} = I, c_{222} = -c_{112} = J$$

The scalars H , I and J are called main scalars and satisfy the equation $H + I = Lc$.

If $C_{\alpha\beta\gamma,\delta}$ are h-scalar derivative of $C_{\alpha\beta\gamma}$, then we have 10

$$c_{0\beta\gamma,\delta} = 0, c_{111,\delta} = H_{,\delta} + 3Jh_{\delta},$$

$$c_{112,\delta} = -J_{,\delta} + (H - 2I)h_{\delta},$$

$$c_{122,\delta} = I_{,\delta} - 3Jh_{\delta},$$

$$C_{222,\delta} = J_{\delta} + 3Ih_{\delta},$$

where h_{δ} are adapted components of h-connection vector h_j .

From (3.2), it follows that

(4.4) $$P_{ijk} = c_{\alpha\beta\gamma,0} e_{(\alpha)i} e_{(\beta)j} e_{(\gamma)k}$$

Since $(\delta_{\alpha\beta} - \delta_{0\alpha}\delta_{0\beta})$ are scalar components of h_{ij} and $c_i = ce_{(1)i}$, therefore from (1.1),(4.2), (4.3) and (4.4), we get

$$H_{,0} + 3Jh_0 = \lambda H + 3\mu c,$$
$$J_{,0} + 3Ih_0 = \lambda J,$$
$$-J_{,0}Hh_0 - 2Ih_0 = -\lambda J,$$
$$I_{,0} - 3Jh_0 = \lambda I + \mu c.$$

Solving these equations, we get

(4.5) $$h_0 = 0, \lambda = \frac{J_{,0}}{J} \, and \, \mu = \frac{I_{,0}}{c} - \frac{J_{,0}I}{cJ}$$

Hence we have the following-

Theorem 4.2: *In a three-dimensional P^* – reducible Finsler space, the scalar component h_0 of h-connection vector h_i vanishes and the scalars λ and μ are given by (4.5).*

Areas of its application: The special Finsler Spaces can be applied in various branches of theory of anisotropic media, lagrangian mechanics, to solve optimization problems, theory of Ecology, Theory of evolution of Biological Systems, theoretical &computational Physics in describing the internal symmetry of Hedrons, theory of Space Time & Gravitation, deformation of crystalline media, Seismic Phenomena, interfaces in thermodynamics system etc.

References

1. M, Matsumoto: The theory of Finsler connection, Publication of the study group of geometry, Okayama University, Japan, (1970).
2. M, Matsumoto: *Foundations of Finsler geometry and special Finsler spaces*, Kaisesieha prees, saikawa, Otsu 520 Japan , 1988.
3. H. Izumi, On P^* –-Finsler spaces I., II. *Memoirs of the Defense Academy, Japan*, **16** (1976) 133–138, **17** (1977) 1–9.
4. H. Izumi., On P^* –-Finsler spaces of scalar curvature, *Tensor, N. S.*, **38** (1982), 220–222.

5. M, Matsumoto, S. Hojo; A conclusive theorem on C-reducible Finsler spaces, Tensor, N. S. **32** (1978), 225–230.

6. P. N. Pandey and Shalini Dikshit; On $^{*}P-$ and P – reducible Finsler Spaces of Recurrent Curvature, *J. Int. Acad. Phys. Sci..*, **2**(1998) 11-17.

7. P. N. Pandey and Reema Verma; On C -Concircularly flat and *P-Finsler spaces, *J. Int. Acad. Phys. Sci..*, **4** (2000) 1-4.

8. P. L. Antonelli, R. S. Ingarden, M, Matsumoto; The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology, Kluwer Acad. Publ., Dordrecht, Boston, London, 1993

9. S. Bácsó, M, Matsumoto; On Finsler spaces of Douglas type, A generalization of the notion of Berwald space, *Publ. Math. Debrecen*, **51** (1997) 385–406.

10. M, Matsumoto; A theory of three dimensional Finsler spaces in terms of scalars. *Demonst. Math.*, **6**(1973) 223-251.