ISSN 0974 - 9373

Vol. 17 No.3 (2013) **Journal of International Academy of Physical Sciences** pp. 255-275

On Quarter-Symmetric Metric Connection in an Indefinite Sasakian Manifold

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(Received July 13, 2013)

Abstract: The object of the present paper is to study a quarter-symmetric metric connection in an indefinite Sasakian manifold. The relation between curvature tensors of quarter-symmetric metric connection and linear connection has been obtained. Also the properties of projective and conformal curvature tensors of quarter-symmetric metric connection in an indefinite Sasakian manifold have been studied.

Key-Words: Indefinite Sasakian manifold, quarter symmetric metric connection, light like Sasakian manifold, time like Sasakian manifold, η-Einstein manifold.

AMS Subject Classification (2000): 53C15.

1. Introduction

The study of manifolds with indefinite metrics is of interest from the standpoint of physics and relativity. Manifolds with indefinite metrics have been studied by several authors. In 1976, Sato[1] introduced an almost paracontact structure on a differentiable manifold, which is an analogue of the almost contact structure[2,3] and is closely related to almost product structure. An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be even-dimensional as well. In 1969, Takahashi[4] studied almost contact manifold equipped with associated pseudo-Riemannian metric. In 1989, Matsumoto[5] replaced the structure vector field ξ by $-\xi$ in an almost paracontact manifold and associated a Lorentzian metric with the resulting structure and called it a Lorentzian almost paracontact manifold. Also in 1993, Bejancu and Duggal[6] introduced the concept of (ε)-Sasakian manifolds and Xufeng and Xiaoli[7] established that these manifolds are real hypersurfaces of indefinite Kaehlerian manifolds. Recently, De and Sarkar[8] introduced (ε)-Kenmotsu manifolds and studied conformally flat, Weyl semi symmetric, ϕ-recurrent (ε)-Kenmotsu manifolds.

On the other hand, the quarter-symmetric connection generalized the semi-symmetric connection. The semi-symmetric connection is important in the geometry of Riemannian manifolds having also physical application; for instance, the displacement on the earth surface following a fixed point is metric and semi-symmetric[9].

In 1975, S. Golab[10] defined and studied quarter-symmetric connection in a differentiable manifold.

A linear connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold (M^n, g) is said to be a quarter-symmetric connection[10] if its torsion tensor $\tilde{T}$ defined by

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y],$$

is of the form

$$\tilde{T}(X,Y) = \eta(Y)\,\phi X - \eta(X)\phi Y, \tag{1.1}$$

where η is 1-form and ϕ is a tensor field of type (1.1). In addition, a quarter-symmetric linear connection $\tilde{\nabla}$ satisfies the condition

$$(\tilde{\nabla}_X g)(Y,Z) = 0, \tag{1.2}$$

for all X, Y, Z $\in$ TM, where TM is the Lie algebra of vector fields of the manifold M^n, then $\tilde{\nabla}$ is said to be a quarter-symmetric metric connection. In particular, if $\phi X = X$ and $\phi Y = Y$, then the quarter-symmetric connection reduces to a semi-symmetric connection[11].

After S. Golab[10], S.C. Rastogi [12,13] continued the systematic study of quarter-symmetric metric connection in a Riemannian, Kaehlerian and Sasakian manifold. In 1982, K. Yano and T. Imai[14] studied quarter-symmetric metric connection in Hermitian and Kaehlerian manifolds. In

1991, S. Mukhopadhyay, A.K. Roy and B. Barua[15] studied quarter-symmetric metric connection on a Riemannian manifold with an almost complex structure ϕ. Quarter-symmetric metric connection are also studied by S.C. Biswas and U.C. De[16], U.C. De and K. De[17], U.C. De and A.K. Mondal[18], R.N. Singh[19], R.N. Singh and S.K. Pandey[20] and many others.

Motivated by the above studies in the present paper, we study an indefinite Sasakian manifold with quarter-symmetric metric connection. In section 2, some preliminary results regarding indefinite Sasakian manifolds are recalled. In section 3, we find the expression for curvature tensor (respectively Ricci tensor) with respect to quarter-symmetric metric connection and investigate relations between curvature tensor (resp. Ricci tensor) with respect to quarter-symmetric metric connection and curvature tensor (respectively Ricci tensor) with respect to Levi-Civita connection. Section 4 deals with projective curvature tensor of M^n with quarter-symmetric metric connection. The last section is devoted to the study of conformal curvature tensor of quarter-symmetric metric connection.

2. Indefinite Sasakian Manifolds

An odd-dimensional semi-Riemannian manifold M^n is called an indefinite almost contact manifold if it admits an indefinite almost contact structure (ϕ, ξ, η) consisting of a (1,1)-tensor field ϕ, a characteristic vector field ξ and a 1-form η satisfying

(2.1) $$\phi^2 X = -X + \eta(X)\xi,$$

(2.2) $$\phi\xi = 0, \ \eta o \phi = 0.$$

If g is a semi-Riemannian metric with (ϕ, ξ, η), that is,

(2.3) $$g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y),$$

(2.4) $$g(X, \xi) = \varepsilon\eta(X), \ g(\xi, \xi) = \varepsilon = \pm 1,$$

(2.5) $$g(X, \phi Y) = -g(\phi X, Y),$$

for all vector fields X and Y on M^n, then (M^n, g) becomes an indefinite almost contact metric manifold equipped with an indefinite almost contact structure (ϕ, ξ, η)[7].

An indefinite almost contact metric manifold M^n is called normal if

(2.6) $$N_\phi + d\eta \otimes \xi = 0,$$

where N_ϕ is the Nijenhuis tensor field.

An indefinite normal contact metric manifold M^n is called an indefinite Sasakian manifold [9] if

(2.7) $$(\nabla_X \eta)(Y) = \varepsilon g(X, \phi Y)$$

(2.8) $$(\nabla_X \xi) = \varepsilon \phi X,$$

(2.9) $$(\nabla_X \phi)(Y) = -g(X, Y)\xi + \varepsilon \eta(Y) X.$$

In view of equation (2.4), ξ is never a light like vector field on M^n. Sasakian manifold with indefinite metrics have been first considered by Takahashi[4]. Their importance for physics has been point out by Duggal [21]. According to the casual character of ξ, we have two classes of Sasakian manifolds. Thus in case ξ is space like ($\varepsilon = 1$ and the index of g is an even number). (respectively, time like, $\varepsilon = -1$ and the index of g is an odd number) we say that M^n is called a space like almost contact metric manifold (respectively, time like almost contact metric manifold)[21].
Also, in an indefinite Sasakian manifold M^n the following relations hold [4].

(2.10) $$\eta(R(X, Y)Z) = \varepsilon [g(X, Z)\eta(Y) - g(Y, Z)\eta(X)],$$

(2.11) $$R(X, Y)\xi = \varepsilon [\eta(Y)X - \eta(X)Y],$$

(2.12) $$R(\xi, X)Y = \varepsilon [g(X, Y)\xi - \eta(Y)X],$$

(2.13) $$S(X, \xi) X = -(n - \varepsilon)\eta(Y),$$

(2.14) $$Q\xi = -(n - \varepsilon)\xi,$$

(2.15) $$S(\phi Y, \phi Z) = S(Y, Z) + \varepsilon (n - \varepsilon)\, \eta(Y)\eta(Z),$$

for any vector fields. X, Y and Z, where R and S are the semi-Riemannian curvature tensor and Ricci tensor respectively.

3. Quarter-Symmetric Metric Connection

Let $\tilde{\nabla}$ be the linear connection and ∇ be the Levi-Civita connection of an indefinite Sasakian manifold M^n such that

(3.1) $$\tilde{\nabla}_X Y = \nabla_X Y + H(X, Y)$$

where H is a tensor field of type (1,1). For $\tilde{\nabla}$ to be a quarter-symmetric metric connection in M^n, we have

(3.2) $$H(X, Y) = \frac{1}{2}\left[\tilde{T}(X, Y) + \tilde{T}'(X, Y) + \tilde{T}'(Y, X)\right]$$

and

(3.3) $$g(T'(X, Y), Z) = g(\tilde{T}(Z, X), Y).$$

In view of equations (1.1) and (3.3), we have

(3.4) $$\tilde{T}'(X, Y) = \eta(X)\phi Y - g(\phi X, Y)\xi.$$

Now, using equations (1.1) and (3.4) in equation (3.2), we get

(3.5) $$H(X, Y) = \eta(Y)\phi X - g(\phi X, Y)\xi.$$

Hence a quarter-symmetric metric connection $\tilde{\nabla}$ in an indefinite Sasakian manifold M^n given by

(3.6) $$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi.$$

Thus the above equation is the relation between quarter-symmetric metric connection and the Levi-Civita connection. The curvature tensor $\tilde{R}$ of M^n with respect to quarter-symmetric metric connection $\tilde{\nabla}$ is defined by

$$\tilde{R}(X,Y)Z=\tilde{\nabla}_X\tilde{\nabla}_X Z-\tilde{\nabla}_Y\tilde{\nabla}_X Z-\tilde{\nabla}_{[X,Y]}Z. \tag{3.7}$$

In view of equation (3.6), above equation takes the form

$$\begin{aligned}\tilde{R}(X,Y)Z=R(X,Y)Z+\varepsilon\eta(Z)\left[\eta(Y)X-\eta(X)Y\right]\\ +\varepsilon[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\xi\\ -\varepsilon\left[g(\phi Y,Z)\,\phi X-g(\phi X,Z)\phi Y\right]\end{aligned} \tag{3.8}$$

where $\tilde{R}$ and R are the curvature tensors of M^n with respect to $\tilde{\nabla}$ and ∇ respectively.

From equation (3.8), it follows that

(3.9)

$$\begin{aligned}{}'\tilde{R}(X,Y,Z,U) = {}'R(X,Y,Z,U)+\varepsilon\eta(Z)\left[\eta(Y)\,g(X,U)-\eta(Y)\,g(Y,U)\right]\\ +\varepsilon\left[g(Y,Z)\,\eta(X)-g(X,Z)\,\eta(Y)\right]\eta(U)\\ -\varepsilon\left[g(\phi Y,Z)\,g(\phi X,U)-g(\phi X,Z)\,g(\phi Y,U)\right].\end{aligned}$$

Let $\{e_i\}_{i=1}^{n}$ be an orthonormal basis of the tangent space at each point of the manifold putting $X=U=e_i$ in equation (3.9) and summing over i, $1\leq i\leq n$, we get

$$\tilde{S}(Y,Z) = S(Y,Z) + \varepsilon(n-\varepsilon)\,\eta(Y)\,\eta(Z), \tag{3.10}$$

which gives

$$\tilde{Q}Y = QY + \varepsilon(n-\varepsilon)\eta(Y)\xi, \tag{3.11}$$

where $\tilde{Q}$ and Q are the Ricci operators of type (1,1), i.e., $S(Y,Z) = g(\tilde{Q}Y,Z)$ and $S(Y,Z) = g(QY,Z)$ with respect to $\tilde{\nabla}$ and ∇ respectively.

Contracting Y in equation (3.11), we get

$$\tilde{r} = r + \varepsilon(n-\varepsilon) \tag{3.12}$$

where $\tilde{r}$ and r are the scalar curvatures with respect to $\tilde{\nabla}$ and ∇ respectively.

Now, writing two more equation by the cyclic permutations of X, Y and Z from equation (3.8), we get

$$\begin{aligned} \tilde{R}(Y,Z)X &= R(Y,Z)X + \varepsilon\eta(X)\left[\eta(Z)Y - \eta(Y)Z\right] \\ &+ \varepsilon\left[g(Z,X)\eta(Y) - g(Y,X)\eta(Z)\right]\xi \\ &- \varepsilon\left[g(\phi Z,X)\phi Y - g(\phi Y,X)\phi Z\right] \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \tilde{R}(Z,X)Y &= R(Z,X)Y + \varepsilon\eta(Y)\left[\eta(X)Z - \eta(Z)Y\right] \\ &+ \varepsilon\left[g(X,Y)\eta(Z) - g(Z,Y)\eta(X)\right]\xi \\ &- \varepsilon\left[g(\phi X,Y)\phi Z - g(\phi Z,Y)\phi X\right]. \end{aligned} \tag{3.14}$$

Adding these two equations to equation (3.8) and using the fact that $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$, we get

$$\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0. \tag{3.15}$$

Thus, we can state as follows

Theorem 3.1 *An indefinite Sasakian manifold admitting a quarter-symmetric metric connection satisfies Bianchi's first identity with respect to* $\tilde{\nabla}$.

Now, interchanging X and Y in equation (3.9) and adding these equation to equation (3.9) with the fact that

$$R(X,Y,Z,U)+R(Y,X,Z,U)=0, \text{ we get}$$

(3.16) $$\tilde{R}(X,Y,Z,U)+\tilde{R}(Y,X,Z,U)=0.$$

Again, interchange Z and U in equation (3.9) and these equation to equation (3.9) with the fact that $R(X,Y,Z,U)+R(X,Y,U,Z)=0$, we get

(3.17) $$\tilde{R}(X,Y,Z,U)+\tilde{R}(X,Y,U,Z)=0.$$

Now, interchanging pair of slots in equation (3.9) and subtracting these equation with the fact that $R(X,Y,Z,U)=R(Z,U,X,Y)$, we get

(3.18) $$\tilde{R}(X,Y,Z,U)=\tilde{R}(Z,U,X,Y).$$

Thus in view of equations (3.16), (3.17) and (3.18), we can state as follows

Theorem 3.2 *In an indefinite Sasakian manifold admitting a quarter-symmetric metric connection, we have*

(i) $\tilde{R}(X,Y,Z,U)+\tilde{R}(Y,X,Z,U)=0,$

(ii) $\tilde{R}(X,Y,Z,U)+\tilde{R}(X,Y,U,Z)=0,$

(iii) $\tilde{R}(X, Y, Z, U) - \tilde{R}(Z, U, X, Y) = 0.$

Now, putting Z = ξ in equation (3.8) and using equations (2.2), (2.4) and (2.11), we get

(3.19)

$$\tilde{R}(X, Y)\xi = (\varepsilon + \varepsilon^2)\,[\eta(Y)X - \eta(X)Y] + \varepsilon^2\,[\eta(Y)X - \eta(X)Y]\xi.$$

Taking the inner product of equation (3.8) with ξ and using equations (2.2), (2.4) and (2.10), we get

(3.20) $$\eta(\tilde{R}(X, Y)Z) = 0.$$

Putting X = ξ in equation (3.8) and using equations (2.2), (2.4) and (2.12), we get

(3.21) $$\tilde{R}(\xi, Y)Z = -\tilde{R}(Y, \xi)Z = \varepsilon(1 + \varepsilon)\,[g(Y, Z)\xi - \eta(Z)Y].$$

By putting Y = ξ in equation (3.10) and use of equations (2.4) and (2.13), we get

(3.22) $$\tilde{S}(X, \xi) = 0.$$

Thus by virtue of equations (3.19), (3.20), 3.21) and (3.22), we can state as follows

Theorem 3.3 *In an indefinite Sasakian manifold admitting quarter-symmetric metric connection, we have*

(i) $\tilde{R}(X, Y)\xi = (\varepsilon + \varepsilon^2)\,[\eta(Y)X - \eta(X)Y] + \varepsilon^2\,[\eta(Y)X - \eta(X)Y]\xi,$

(ii) $\eta(R(X, Y)Z) = 0,$

(iii) $\tilde{R}(\xi, Y)Z = \varepsilon(1 + \varepsilon)\,[g(Y, Z)\xi - \eta(Z)Y],$

(iv) $\tilde{S}(X,\xi) = 0.$

Now, consider

(3.23) $R(\xi,X).\,\tilde{R}(Y,Z)U = 0,$

which gives

$$R(\xi,X)\tilde{R}(Y,Z)U - \tilde{R}(R(\xi,X)Y,Z)U - \tilde{R}(Y,R(\xi,X)Z)U - \tilde{R}(Y,Z)R(\xi,X)Z = 0. \tag{3.24}$$

In view of equations (2.12), (3.19) and (3.20), above equation takes the form

(3.25)

$$\varepsilon\Big[g(X,\tilde{R}(Y,Z)U)\xi - g(X,Y)R(\xi,Z)U + \eta(Y)\tilde{R}(X,Z)U - g(X,Z)\tilde{R}(Y,\xi)U + \eta(Z)\tilde{R}(Y,X)U - g(X,U)\tilde{R}(Y,Z)\xi + \eta(U)\tilde{R}(Y,Z)X\Big] = 0.$$

By virtue of equation (3.8), above equation reduces to

$$\begin{aligned}
(3.26)\quad & g(X,R(Y,Z)U)\xi + \varepsilon\eta(U)\left[g(X,Y)\eta(Z) - g(X,Z)\eta(Y)\right]\xi \\
& + \varepsilon^2\eta(X)\left[g(Z,U)\eta(Y) - g(Y,U)\eta(Z)\right]\xi \\
& - \varepsilon\left[g(\varphi Z,U)\,g(X,\varphi Y) - g(\varphi Y,U)\,g(X,\varphi Z)\right]\xi \\
& - \varepsilon(1+\varepsilon)\left[g(X,Y)\,g(Z,U)\xi - g(X,Y)\eta(U)Z\right] \\
& + \varepsilon\eta(Y)\eta(U)\left[\eta(Z)X - \eta(X)Z\right] \\
& + \varepsilon\eta(Y)\left[g(Z,U)\eta(X) - g(X,U)\eta(Z)\right]\xi \\
& - \varepsilon\eta(Y)\left[g(\varphi Z,U)\varphi X - g(\varphi X,U)\varphi Z\right]
\end{aligned}$$

$$+\varepsilon(1+\varepsilon)[g(X,Z)g(Y,U)\xi-g(X,Z)\eta(U)Y]$$
$$+\varepsilon\eta(Z)\eta(U)[\eta(X)Y-\eta(Y)X]$$

$$+\varepsilon\eta(Z))[g(X,U)\eta(Y)-g(Y,U)\eta(X)]\xi$$
$$-\varepsilon\eta(Z)\,[g(\varphi X,U)\varphi Y-g(\varphi Y,U)\varphi X]$$

$$-\varepsilon(1+\varepsilon)[g(X,U)\eta(Z)Y-g(X,U)\eta(Y)Z]$$
$$-\varepsilon^2[g(X,U)\eta(Z)Y-g(X,U)\eta(Y)Z]\xi$$

$$+\varepsilon\eta(U)\eta(X)[\eta(Z)Y-\eta(Y)Z]$$
$$+\varepsilon\eta(U)[g(Z,X)\eta(Y)-g(Y,X)\eta(Z)]\xi$$

$$-\varepsilon\eta(U)[g(\varphi Z,X)\varphi Y-g(\varphi Y,X)\varphi Z]$$
$$+\eta(Y)R(X,Z)U+\eta(Z)R(Y,X)U$$

$$+\eta(U)R(Y,Z)X=0.$$

Taking the inner product of above equation with ξ and using equations (2.2), (2.4) and (2.10), we get

$$\varepsilon'R(Y,Z,U,X)+\varepsilon^2\eta(U)[g(X,Y)\eta(Z)-g(X,Z)\eta(Y)] \tag{3.27}$$

$$+\varepsilon^2\eta((X)[g(Z,U)\eta(Y)-g(Y,U)\eta(Z)]$$

$$-\varepsilon^2[g(\phi Z,U)g(X,\phi Y)-g(\phi Y,U)g(X,\phi Z)]$$

$$-\varepsilon^2(1+\varepsilon)[g(X,Y)g(Z,U)-g(X,Y)\eta(U)\eta(Z)]$$

$$+\varepsilon^2(1+\varepsilon)[g(X,Z)g(Y,U)-g(X,Z)\eta(Y)\eta(U)]=0.$$

Putting Y = X = e_i in above equation and taking summation over $i, 1 \leq i \leq n$ we get

(3.28) $$S(Z,U) = [\varepsilon(1+\varepsilon)(\eta-1)+\varepsilon(1-\varepsilon)]g(Z,U) + [\varepsilon(1+\varepsilon)-(n-1)\varepsilon-2n]\eta(Z)\eta(U).$$

For $\varepsilon = 1,$ above equation reduces to

(3.29) $$S(Z,U) = 2(n-1)g(Z,U)+3(1-n)\eta(Z)\eta(U).$$

This shows that M^n is an η-Einstein manifold.

Thus we can state as follows

Theorem 3.4 *A space like indefinite Sasakian manifold* M^n *with quarter- symmetric metric connection satisfying* $R(\xi, X).\tilde{R}(Y,Z)U = 0,$ *is an η-Einstein manifold.*

Again, for $\varepsilon = 1,$ equation (3.28) takes the form

(3.30) $$S(Z,U) = -2\,g(Z,U) + (-(n+1))\eta(Z)\eta(U),$$

This shows that M^n is an η-Einstein manifold.

Thus, we can state as follows

Theorem 3.5 *A time like indefinite Sasakian manifold* M^n *with quarter-symmetric metric connection satisfying* $R(\xi, X).\tilde{R}(Y,Z)U = 0,$ *is an η-Einstein manifold.*

4. Projective Curvature Tensor of Quarter-Symmetric Metric Connection

Projective curvature tensor of quarter-symmetric metric connection $\tilde{\nabla}$ in M^n is defined as

$$(4.1)\quad \tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{(n-1)}\left[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\right],$$

which on using equations (3.8) and (3.10), gives

$$(4.2)\quad \tilde{P}(X,Y)Z = R(X,Y)Z + \frac{\varepsilon(\varepsilon-1)}{(n-1)}\eta(Z)\left[\eta(Y)X - \eta(X)Y\right]$$

$$+\varepsilon\left[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\right]\xi - \varepsilon\left[g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y\right]$$

$$-\frac{1}{(n-1)}\left[S(Y,Z)X - S(X,Z)Y\right],$$

which gives

$$(4.3)\quad \tilde{P}(X,Y)Z = P(X,Y)Z + \frac{\varepsilon(\varepsilon-1)}{(n-1)}\eta(Z)\left[\eta(Y)X - \eta(X)Y\right]$$

$$+\varepsilon\left[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\right]\xi - \varepsilon\left[g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y\right],$$

where P(X,Y)Z is the projective curvature tensor of connection ∇ [23] define as

$$(4.4)\quad P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}\left[S(Y,Z)X - S(X,Z)Y\right],$$

Putting $X = \xi$ in equation (4.2) and using equations (2.2), (2.4), (2.12) and (2.13), we get

$$(4.5)\qquad \tilde{P}(\xi, Y)Z = -\tilde{P}(Y,\xi)Z = (1+\varepsilon)\left[g(Y,Z)\xi - \eta(Z)Y\right]$$

$$+\left(\frac{2-n-6}{n-1}\right)\eta(Y)\eta(Z)\xi - \frac{1}{(n-1)}S(Y,Z)\xi,$$

Again putting Z = ξ in equations (4.2) and using equations (2.2), (2.4), (2.11) and (2.13), we get

$$(4.6)\qquad \tilde{P}(X,Y)\xi = (1+\varepsilon)\left[\eta(Y)X - \eta(X)Y\right].$$

Now, taking the inner product of equation (4.2) with ξ and using equations (2.2), (2.4) and (2.10), we get

$$(4.7)\qquad \eta(\tilde{P}(X,Y)Z) = -\frac{\varepsilon}{(n-1)}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)].$$

Thus in view of equations (4.5), (4.6) and (4.7), we can state as follows

Theorem 4.1 *In an indefinite Sasakian manifold with quarter-symmetric metric connection, we have*

(i) $\tilde{P}(\xi, Y)Z = (1+\varepsilon)\left[g(Y,Z)\xi - \eta(Z)Y\right]$

$$+\left(\frac{2-n-\varepsilon}{n-1}\right)\eta(Y)\eta(Z)\xi - \frac{1}{(n-1)}S(Y,Z)\xi,$$

(ii) $\tilde{P}(X,Y)\xi = (1+\varepsilon)\left[\eta(Y)X - \eta(X)Y\right]$

(iii) $\eta(\tilde{P}(X,Y)Z) = -\dfrac{\varepsilon}{(n-1)}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)].$

Now, suppose $R(\xi, X).\,\tilde{P}(Y,Z)U = 0$, in an indefinite Sasakian manifold, then we have

$$
\begin{aligned}
&R(\xi,X)\tilde{P}(Y,Z)U)-\tilde{P}(R(\xi,X)Y,Z)U \\
&-\tilde{P}(Y,R(\xi,X)Z)U-\tilde{P}(Y,Z)R(\xi,X)U=0.
\end{aligned}
\tag{4.8}
$$

By virtue of equation (2.12), above equation takes the form

$$
\varepsilon\left[\begin{aligned}
&g(X,\tilde{P}(Y,Z)U)\xi-\eta(\tilde{P}(Y,Z)U)X \\
&-g(X,Y)\tilde{P}(\xi,Z)U+\eta(Y)\tilde{P}(X,Z)U \\
&-g(X,Z)\tilde{P}(Y,\xi)U+\eta(Z)\tilde{P}(Y,X)U \\
&-g(X,U)\tilde{P}(Y,Z)\xi+\eta(U)\tilde{P}(Y,Z)X
\end{aligned}\right]=0.
\tag{4.9}
$$

Taking the inner product of above equation with ξ and using equation (2.4), we get

$$
\begin{aligned}
&\varepsilon\big[g(X,\tilde{P}(Y,Z)U)-\eta(\tilde{P}(Y,Z)U)\eta(X)-g(X,Y)\eta(\tilde{P}(\xi,Z)U) \\
&+\eta(Y)\eta(\tilde{P}(X,Z)U)-g(X,Z)\eta(\tilde{P}(Y,\xi)U)+\eta(Z)\eta(\tilde{P}(Y,X)U) \\
&-g(X,U)\eta(\tilde{P}(Y,Z)\xi)+\eta(U)\eta(\tilde{P}(Y,Z)X)\big]=0.
\end{aligned}
\tag{4.10}
$$

In view of equations (4.2), (4.5), (4.6) and (4.7), above equation takes the form

$$
\begin{aligned}
\varepsilon\Big[&{}'R(Y,Z,U,X)+\frac{\varepsilon(\varepsilon-1)}{(n-1)}\eta(U)\{g(X,Y)\eta(Z)-g(X,Z)\eta(Y)\} \\
&+\varepsilon\eta(X)\{g(Z,U)\eta(Y)-g(Y,U)\eta(Z)\} \\
&-\varepsilon\{g(\varphi Z,U)g(X,\varphi Y)+g(Y,\varphi U)g(X,\varphi Z\} \\
&-\frac{\varepsilon}{(n-1)}\eta(U)\{S(Z,X)\eta(Y)-S(X,Y)\eta(Z)\}
\end{aligned}
\tag{4.11}
$$

$$+(1+\varepsilon)\begin{Bmatrix} g(X,Y)\,g(Z,U)-g(X,Y)\eta(Z)\eta(U) \\ -g(X,Z)\,g(Y,U)+g(X,Z)\eta(Y)\eta(U) \end{Bmatrix}$$

$$\left.-\frac{(2-n-\varepsilon)}{(n-1)}\{g(X,Y)\eta(Z)\eta(U)-g(X,Z)\eta(Y)\eta(U)\}\right]=0.$$

Putting X = Y = e_i in above equation and taking summation over $i, 1 \le i \le n$, we get

$$(4.1)\quad S(Z,U)=(1-\eta)(1+\varepsilon)\,g(Z,U)+\frac{[(\varepsilon-n)(\varepsilon^2-n\varepsilon-1)-\varepsilon r]}{(n-1)}\eta(Z)\eta(U)$$

For $\varepsilon=1$, above equation takes the form

$$(4.13)\quad S(Z,U)=2(1-n)\,g(Z,U)+\frac{[n(n-1)-r]}{(n-1)}\eta(Z)\eta(U).$$

This shows that M^n is an η-Einstein manifold.

Thus we can state as follows

Theorem 4.2 *A space like indefinite Sasakian manifold M^n with quarter-symmetric metric connection satisfying* $R(\xi,X).\tilde{P}(Y,Z)U=0$, *is an η-Einstein manifold.*

Again for $\varepsilon=-1$, equation (4.12), takes the form

$$(4.14)\qquad S(Z,U)=\frac{r-n(n+1)}{(n-1)}\eta(Z)\eta(U).$$

Thus, we can state as follows

Theorem 4.3 *On a time like indefinite Sasakian manifold M^n with quarter symmetric metric connection, if* $R(\xi, X).\,\tilde{P}(Y,Z)U = 0$, *then equation (4.14) holds.*

5. Conformal Curvature Tensor of Quarter-Symmetric Metric Connection

Conformal curvature tensor $\tilde{C}$ of quarter-symmetric metric connection $\tilde{\nabla}$ in M^n is defined as

$$(5.1)\quad \tilde{C}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{(n-2)}\begin{bmatrix} \tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y \\ +g(Y,Z)\tilde{Q}X - g(X,Z)\tilde{Q}Y \end{bmatrix}$$

$$+\frac{\tilde{r}}{(n-1)(n-2)}\left[g(Y,Z)X - g(X,Z)Y\right].$$

Using equations (3.8), (3.10) (3.11) and (3.12) in above equation, we get

$$(5.2)\,\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}\begin{bmatrix} S(Y,Z)X - S(X,Z)Y + \\ g(Y,Z)QX - g(X,Z)QY \end{bmatrix}$$

$$+\frac{r}{(n-1)(n-2)}\left[g(Y,Z)X - g(X,Z)Y\right] - \varepsilon\left[g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y\right]$$

$$+\frac{\varepsilon(\varepsilon-2)}{(n-2)}\eta(Z)\left[\eta(Y)X - \eta(X)Y\right]$$

$$+\frac{\varepsilon(\varepsilon-2)}{(n-2)}\left[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\right]\xi$$

$$+\frac{\varepsilon(n-\varepsilon)}{(n-1)(n-2)}\left[g(Y,Z)X-g(X,Z)Y\right],$$

which yields

$$\text{(5.3)}\quad \tilde{C}(X,Y)Z = C(X,Y)Z-\varepsilon\left[g(\phi Y,Z)\phi X-g(\phi X,Z)\phi Y\right]$$

$$+\frac{\varepsilon(\varepsilon-2)}{(n-2)}\eta(Z)\left[\eta(Y)X-\eta(X)Y\right]$$

$$+\frac{\varepsilon(\varepsilon-2)}{(n-2)}\left[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\right]\xi$$

$$+\frac{\varepsilon(n-\varepsilon)}{(n-1)(n-2)}\left[g(Y,Z)X-g(X,Z)Y\right],$$

where C(X,Y)Z is the conformal curvature tensor of connection ∇ in M^n [23] defined as

$$\text{(5.4)}\, C(X,Y)Z = R(X,Y)Z-\frac{1}{(n-2)}\begin{bmatrix} S(Y,Z)X-S(X,Z)Y \\ +g(Y,Z)QX-g(X,Z)QY \end{bmatrix}$$

$$+\frac{r}{(n-1)(n-2)}\left[g(Y,Z)X-g(X,Z)Y)\right].$$

Now interchanging X and Y in equation (5.2) and adding these equation to equation (5.2) with the fact that $R(X,Y)Z+R(Y,X)Z=0$, we get

$$\text{(5.5)}\qquad \tilde{C}(X,Y)Z+\tilde{C}(Y,X)Z=0.$$

Thus, we can state as follows

Theorem (5.1): *In an indefinite Sasakian manifold M^n with quarter-symmetric metric connection, we have*

$$\tilde{C}(X, Y)Z + \tilde{C}(Y, X)Z = 0.$$

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