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A Least Cost Assignment Technique for Solving Assignment Problems

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Abstract: Assignment problem is a special kind of transportation problem in which each source should have the capacity to fulfill the demand of any of the destination. In other words, any operator should be able to perform any job regardless of his skills. Classical Assignment Problem (AP) is a well-known topic discussed in real physical world. We consider many classical problems from location theory which may serve as theoretical models for several logistic problems such that some linear or quadratic function attains its minimum. It turns out that linear objective function yields a linear assignment problem, which can be solved easily by several primal-dual methods like Hungarian method, Shortest augmenting path method etc. In this paper, a new approach is applied to solve all types of assignment problems with minimized or maximized objective functions, namely least cost assignment technique. This method proposes momentous advantages over similar methods. Proposed method is a systematic procedure and easy to apply.

Keywords: Assignment Problem, Least Cost Assignment Technique.
AMS Classification: 90Bxx

1. Introduction

The assignment problem is one of the earliest applications of linear integer programming problem. A variety of practical problems turn out to be a special instance of the assigning problem, i.e. a problem, where one looks for an assignment of members of set A to members of set B such that some function attains its optimum. It may always assumed that the numbers of elements in sets A and B are equal and that we want to assign exactly one element from A to each element from B. Solving such a problem often means that some functions are evaluated for each assignment. Since there are n! possible assignments, where n is the number of elements in A, this may lead to a very hard problem. The assignment problem arises in a variety of situations. Assignment problem is a typical combinatorial optimization problem, which are

i. Assigning jobs to machines
ii. Assigning sales personnel to sales territories
iii. Assigning contracts to bidders
iv. Assigning agent to tasks
v. Assigning taxis to customers

The distinguishing feature of assignment problems is that one agent is assigned to one and only one tasks. For example, if we have three taxies A, B, C and three customers 1, 2, 3, who want a taxi then A→3, B→1, C→2 is a possible assignment.

We will initially assume that

Number of agents = Number of tasks.

Problem satisfying this condition are called balanced problem.

Example: Suppose that some company wants to build a factory (building 1) to produce certain products, a warehouse (building 2) to store the products and a new shop (building 3) to sell these products. It owns three pieces of land in three different cities. It wants to locate these buildings on the given sites of land in such a way that

i. The total construction cost plus
ii. The total interaction between these buildings is minimal.

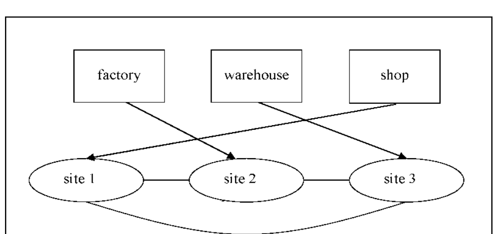

Figure1. Example of an assignment problem with 3 buildings and 3 sites

Different methods have been presented to solve assignment problem by flood (1956), Gupta (2008), Wayne (2010), Hadi (2012). A considerable number of methods have been so far presented for assignment problem in which, the best known, most widely used, and most written about method for solving the assignment problem is the Hungarian Method, originally suggested by Kuhn in 1955. Ford and Fulkerson (1955) provided vital ideas for the untimely approaches used in solving network flow problems, which extended to solve the transportation problem, and generalized to solve the linear programming problem. It is a dual method with a feasible assignment being obtained only at the last computational step.

In this paper, least cost assignment technique is applied for all type of assignment problems. The proposed method is a methodical process, easy to apply and can be exploited for all types of assignment problems with maximize or minimize objective functions presented through numerical examples.

2. Mathematical Formulation of Assignment Problem

Assignment problem is a special kind of transportation problem in which each source should have the capacity to fulfil the demand of any of the destinations. In other words, any operator should be able to perform any job regardless of his skills, although the cost will be more if the job does not match with the workers skill. An example of assigning operators to jobs in a shop floor situation is shown through Table-1.

Operators → jobs ↓	**1**	**2**	**3**	**………. J**	**………..n**
1	c_{11}	c_{12}	c_{13}	………. c_{1j}	……….. c_{1n}
2	c_{21}	c_{22}	c_{23}	………. c_{2j}	……….. c_{2n}
3	c_{31}	c_{32}	c_{33}	………. c_{3j}	……….. c_{3n}
……….. ……….. i	…… …… c_{i1}	….. ….. c_{i2}	….. ….. c_{i3}	………… ………… ……….. c_{ij}	………… ………… ………… c_{in}
……….. ……….. n	….. ….. c_{n1}	….. ….. c_{n2}	…. …. c_{n3}	……….. ……….. ……….. c_{nj}	…………. …………. …………. c_{nn}

Table1. Generalized Format of Assignment Problem

Here, m is the numbers of jobs as well as the number of operators and c_{ij} be the processing time of the job i if it is assigned to the operator j. The objective is to assign the jobs to the operators such that the total processing time is minimized. Table 2 summarized different examples of the assignment problems.

Row Entry	**Column Entry**	**Cell Entry**
Jobs	Operators	Processing time
Operators	Machine	Processing time
Teachers	Subjects	Student pass percentage
Physicians	Treatments	Number of case handled
Drivers	Routs	Travel time

Table 2. Examples of Assignment problem

Zero-One Programming Model for Assignment Problem

A zero-one programming model for assignment problem is presented below

$$Minimize\ Z = c_{11}x_{11} + c_{12}x_{12} + ... + c_{1m}x_{1m} + c_{21}x_{21} + c_{22}x_{22} + ... + c_{2m}x_{2m} + ...$$
$$+ c_{i1}x_{i1} + c_{i2}x_{i2} + ... + c_{im}x_{im} + ... + c_{m1}x_{m1} + c_{m2}x_{m2} + ... + c_{mm}\,x_{mm}.$$

Subject to $x_{11} + x_{12} + ... + x_{1j} + ... + x_{1m} = 1$,

$$x_{21} + x_{22} + ... + x_{2j} + ... + x_{2m} = 1,$$
$$\vdots$$
$$x_{i1} + x_{i2} + ... + x_{ij} + ... + x_{im} = 1,$$
$$\vdots$$
$$x_{m1} + x_{m2} + ... + x_{mj} + ... + x_{mm} = 1,$$
$$x_{11} + x_{21} + ... + x_{i1} + ... + x_{m1} = 1,$$
$$x_{12} + x_{22} + ... + x_{i2} + ... + x_{m2} = 1,$$
$$\vdots$$
$$x_{1j} + x_{2j} + ... + x_{ij} + ... + x_{mj} = 1,$$
$$\vdots$$

(2.1) $$x_{1m} + x_{2m} + ... + x_{im} + ... + x_{mm} = 1,$$

$x_{ij} = 0$ or 1, for $i = 1, 2, ..., m$ and

$j = 1, 2, ..., m.$

The above model is presented in a short form as

(2.2) $$Min\ z = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}x_{ij}$$

(2.3) $$\text{subject to}\begin{cases} \sum_{j=1}^{m} x_{ij} = 1, & i = 1,2,....,m \\ \sum_{i=1}^{m} x_{ij} = 1, & j = 1,2,....,\text{m} \\ x_{ij} = 0 \text{ or } 1, & i, j = 1,2,...,\text{m} \end{cases}$$

here, m being the number of rows (jobs) as well as the number of columns (operators) and c_{ij} the time/cost of assigning the row i to the column j.

Thus, $$x_{ij} = 1, \text{ if the row } i \text{ is assigned to the column } j$$
$$= 0, \text{ otherwise.}$$

In this model the objective function minimizes the total cost of assigning the row to the columns. The first set of constraints ensures that each row is assigned to only one column. The second set of constraints ensures that each column is assigned to only one row.
Here c_{ij} can be replaced by d_{ij} in the zero-one assignment problem in the maximization form and can solve it by any standard procedure (Hungarian method or by any software) to get the optimal assignment. In that case the model for the preference assignment problem becomes

(2.4) $$Max\ z = \sum_{i=1}^{m}\sum_{j=1}^{m} d_{ij}x_{ij}$$

(2.5) $$\text{subject to}\begin{cases} \sum_{i=1}^{n} x_{ij} = 1, & j = 1,2,....,\text{m} \\ \sum_{j=1}^{n} x_{ij} = 1, & \text{i} = 1,2,....,\text{m} \\ x_{ij} = 0 \text{ or } 1, & i, j = 1,2,...,\text{m} \end{cases}$$

3. Method of Solution

Algorithm for finding the optimal solution of assignment problem is presented below

Step-I. First, we find row penalties, i.e. the difference between the first minimum and second minimum in each row. If the two minimum values are same then the row penalty is zero.

Step-II. Find column penalties, i.e. the difference between the first minimum and second minimum in each column. If the two minimum values are same then the column penalty is zero.

Step-III. Find the maximum penalty amongst the row penalties and the column penalties and identify whether it occurs in a row or in a column. If the maximum penalties is in a row then go to next step otherwise go to step-VI.

Step-IV. Identify the cell for assign which has the least cost in that row.

Step-V. Assign this least cost to that cell and go to step-IX.

Step-VI. Identify the cell for assign which has the least cost in that column.

Step-VII. Find the least cost in that cell.

Step-VIII. Now assign this least value to the selected cell.

Step-IX. Check whether exactly one of the row/column corresponding to the selected cell. If yes, than go to next step.

Step-X. Delete both the row and column which has the assign element to the selected cell. Then revise the above procedure for row and column penalties.

Step-XI. Check whether exactly one row/column is left out, if yes then we get the complete solution.

Step-XII. At last we make all assign elements in given table, which is optimal solution of given problem.

Types of Assignment Problems

There are four types of assignment problems

Type-I: Balanced assignment problem- If the number of rows (jobs) is equal to the number of column (operators), then the problem is termed as a balanced assignment problem.

Type-II: Unbalanced assignment problem- Whenever the cost matrix of an assignment problem is not a square matrix is called an unbalanced assignment problem. If the problem is unbalanced, like an unbalanced transportation problem, then necessary numbers of dummy row(s)/column(s) are added such that the cost matrix is a square matrix the values for the entries in the dummy row(s)/column(s) are assumed to be zero, under such a condition, while implementing the solution, the dummy row(s) or column(s) will not have assignment(s).

Type-III: Maximization case in assignment problems- There are problems where certain facilities have to be assigned to a number of jobs so as to maximize the overall performance of the assignment. The problem can be converted into a minimization problem in the following two ways

(a) select the greatest element of the given cost matrix and then subtract each element of the matrix from the greatest element to get the modified matrix.

(b) Multiply each element of the matrix by (-1) to get the modified matrix.

Type-IV: Problem with restricted (infeasible) assignment- It is sometimes possible that a particular person is in capable of doing curtain work or a specific job cannot be performed on a particular machine. The solution of the assignment problem should take into account these restrictions so that the infeasible assignment can be avoided. This can be achieved by assigning a very high cost M (say) to the sales where assignments are prohibited, there by restricting the entry of this pair of job-machine into the final solution. Some cases a certain worker cannot be assigned a particular job. The reasons for impossible assignments are numerous. Back of required skills, deficiency in technical know-how, improper training and physical inability are only a few many reasons. For solving such type of assignment problems we put infinite cost in the cell where no assignment is possible. The remaining procedure is exactly the same as the ordinary assignment problems.

3. Numerical Examples

Example–1. A department head has four subordinates and four tasks to be performed. The subordinates differ in efficiency and the task differs in their intrinsic difficulty. His estimate of the times each man would take to perform each task is given in the effectiveness matrix below. How should the task be allocated, one to a man, so as to minimize the total man hour

Tasks	Subordinates			
↓	I	II	III	IV

A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

Table 3

Solution:

Step-I. First, we calculate the penalties for all rows and columns. They are written below the Table 4 for the row difference and aside the table for the column differences. We now select column, III because it has the highest penalty rating 13.we look at the cell with the least cost in the column III is to be identified. Here it is the cell (B, III) with entry 4. So we assign 4 to (B, III). Hence row and column are deleted corresponding to cell (B, III) and resulting data shown in Table 5

Tasks ↓	**Subordinates**				**Penalty** ↓
	I	II	III	IV	
A	8	26	17	11	3
B	13	28	$\boxed{4}$	26	9
C	38	19	18	15	3
D	19	26	24	10	9
Penalty	5	7	13	1	

Table 4

Step-II. We now select column I because it has the highest penalty rating 11.we look at the cell with the least cost in the column I is to be identified. Here it is the cell (A, I) with entry 8. So we assign 8 to (A, 1). Hence row and column are deleted corresponding to cell (A, I) and resulting data shown in Table 6

Tasks ↓	**Subordinates**			**Penalty** ↓
	I	II	IV	
A	$\boxed{8}$	26	11	3
C	38	19	15	4
D	19	26	10	9
Penalty	11	7	1	

Table 5

Step-III. We now select row D because it has the highest penalty rating 16.we look at the cell with the least cost in the row D is to be identified. Here it is the cell (D, II) with entry 10. So we assign 10 to (D, II). Hence row and column are deleted corresponding to cell (D, II) and resulting data shown in Table 7

Tasks	Subordinates		Penalty
↓	I	II	↓
C	19	15	4
D	26	$\boxed{10}$	16
Penalty	7	5	

Table 6

Step-IV. Here we have only one cost so we assign that cost, which is shown in Table 7.

Tasks	Subordinates
↓	I
C	$\boxed{19}$

Table 7

Step-V. The whole procedure can be done in a single table as given below

Tasks	Subordinates			
↓	I	II	III	IV
A	$\boxed{8}$	26	17	11
B	13	28	$\boxed{4}$	26
C	38	$\boxed{19}$	18	15
D	19	26	24	$\boxed{10}$

Table 8

Hence optimal solution is: A→I, B→III, C→II, D→IV
Optimal value = 8+4+19+10=41.

Example-2. A company has four machines to do three jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table

Jobs	Machine			
↓	W	X	Y	Z
A	18	24	28	32
B	8	13	17	19
C	10	15	19	22

Table 9

What are the job assignments which will minimize the cost?

Solution: **Step-I**. This is a case of unbalanced assignment problem. So we introduce a fictitious (dummy) job D with all entries zero and after this, the technique is applied.

Jobs	Machine			
↓	W	X	Y	Z
A	18	24	28	32

B	8	13	17	19
C	10	15	19	22
D	0	0	0	0

Table 10

Step-II. First, we calculate the penalties for all rows and columns. They are written below the table for the row difference and aside the table for the column differences. We now select column Z because it has the highest penalty rating 19.we look at the cell with the least cost in the column Z is to be identified. Here it is the cell (D, Z) with entry 0. So we assign 0 to (D, Z). Hence row and column are deleted corresponding to cell (D, Z) and resulting data shown in Table 12.

Jobs	Machine				Penalty
↓	W	X	Y	Z	↓
A	18	24	28	32	6
B	8	13	17	19	5
C	10	15	19	22	5
D	0	0	0	$\boxed{0}$	0
Penalty	8	13	17	19	

Table 11

Step-III. We now select row A because it has the highest penalty rating 6.we look at the cell with the least cost in the row A is to be identified. Here it is the cell (A, W) with entry 18. So we assign 18 to (A, W). Hence row and column are deleted corresponding to cell (A, W) and resulting data shown in Table 13.

Jobs	Machine			Penalty
↓	W	X	Y	↓
A	$\boxed{18}$	24	28	6
B	8	13	17	5
C	10	15	19	5
Penalty	2	2	2	

Table 12

Step-IV. We now select row B because it has the highest penalty rating 4.we look at the cell with the least cost in the row B is to be identified. Here it is the cell (B, X) with entry 13. So we assign 13 to (B, X). Hence row and column are deleted corresponding to cell (B, X) and resulting data shown in Table 14

Jobs ↓	Machine X	Machine Y	Penalty ↓
B	$\boxed{13}$	17	4
C	15	19	4
Penalty	2	2	

Table 13

Step-V. Here we have only one cost so we assign that cost, which is shown in Table 14.

Jobs ↓	Machine Y
C	$\boxed{19}$

Table 14

Step-VI. The whole procedure can be done in a single table as the given below.

Jobs ↓	Machine W	X	Y	Z
A	$\boxed{18}$	24	28	32
B	8	$\boxed{13}$	17	19
C	10	15	$\boxed{19}$	22
D	0	0	0	$\boxed{0}$

Table 15

Hence optimal solution is: A→W, B→X, C→Y, D→Z
The minimum cost = 18+13+19+0=50 Rs.

Example-3. A marketing manner has five salesman and sales-districts. Considering the capabilities of the salesman and the nature of districts, the marketing manager estimates that
sale per month (in hundred Rupees) for each salesman in each district would be as follows

Salesman ↓	Districts A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

Table 16

Find the assignment of salesman to districts that will result in maximum sales.

Solution:

Step-I. This is an assignment problem of profit maximization type; we convert it into an assignment problem of cost minimization type by formatting anew modified matrix, obtained by subtracting each entry of the matrix from the greatest entry 41 of the matrix. The resulting problem will solve as the problem of minimization cost.

Salesman	Districts				
↓	A	B	C	D	E
1	9	3	1	13	1
2	1	17	13	20	5
3	0	14	8	11	4
4	19	3	0	5	5
5	12	11	1	6	2

Table 17

Step-II. First, we calculate the penalties for all rows and columns. They are written below the table for the row difference and aside the table for the column differences. We now select row 3 because it has the highest penalty rating 4.we look at the cell with the least cost in the row 3 is to be identified. Here it is the cell (3, A) with entry 0. So we assign 0 to (3, A). Hence row and column are deleted corresponding to cell (3, A) and resulting data shown in Table 19.

Salesman	Districts					Penalty
↓	A	B	C	D	E	↓
1	9	3	1	13	1	0
2	1	17	13	20	5	4
3	$\boxed{0}$	14	8	11	4	4
4	19	3	0	5	5	3
5	12	11	1	6	2	1
Penalty	1	0	1	1	1	

Table 18

Step-III: We now select row 2 because it has the highest penalty rating 8.we look at the cell with the least cost in the row 2 is to be identified. Here it is the cell (2, E) with entry 5. So we assign 5 to (2, E). Hence row and column are deleted corresponding to cell (2, E) and resulting data shown in Table 20.

Salesman	Districts				Penalty
↓	B	C	D	E	↓
1	3	1	13	1	0
2	17	13	20	$\boxed{5}$	8
4	3	0	5	5	5
5	11	1	6	2	1
Penalty	0	1	1	1	

Table 19

Step-IV: We now select row 5 because it has the highest penalty rating 5.we look at the cell with the least cost in the row 5 is to be identified. Here it is the cell (5, C) with entry 1. So we assign 1 to (5, C). Hence row and column are deleted corresponding to cell (5, C) and resulting data shown in Table 21

Salesman ↓	Districts B	C	D	Penalty ↓
1	3	1	13	2
4	3	0	5	3
5	11	$\boxed{1}$	6	5
Penalty	0	1	1	

Table 20

Step-V: We now select row 1 because it has the highest penalty rating 10.we look at the cell with the least cost in the row 1 is to be identified. Here it is the cell (1, B) with entry 3. So we assign 3 to (1, B). Hence row and column are deleted corresponding to cell (1, B) and resulting data shown in Table 22

Salesman ↓	Districts B	D	Penalty ↓
1	$\boxed{3}$	13	10
4	3	5	2
Penalty	0	8	

Table 21

Step-VI. Here we have only one cost so we assign 5 to (4, D), which is shown in Table 23.

Salesman ↓	Districts D
4	$\boxed{5}$

Table 22

Step-VII. The whole procedure can be done in a single table as given below

Salesman ↓	Districts A	B	C	D	E
1	32	$\boxed{38}$	40	28	40
2	40	24	28	21	$\boxed{36}$
3	$\boxed{41}$	27	33	30	37
4	22	38	41	$\boxed{36}$	36
5	29	33	$\boxed{40}$	35	39

Table 23

Hence optimal solution is: 1→B, 2→E, 3→A, 4→D, 5→C, The maximum sales = 38+36+41+36+40=191×100=19100 Rs.

Example-4. The secretary of a school is taking bids on the city's four school bus routes. Four companies have made the bids a detailed in the following table

Company ↓	Bids			
	Route 1	Route 2	Route 3	Route 4
1	4	5	-	-
2	-	4	-	4
3	3	-	2	-
4	-	-	4	5

Table 24

Suppose each bidder can be assigned only one route. Use the assignment model to minimize the school's cost of running the four routes.

Solution:

Step-I. First, we calculate the penalties for all rows and columns. They are written below the table for the row difference and aside the table for the column differences. We now select column 3 because it has the highest penalty rating 2.we look at the cell with the least cost in the column3 is to be identified. Here it is the cell (3, 3) with entry 2. So we assign 2 to (3, 3). Hence row and column are deleted corresponding to cell (3, 3) and resulting data shown in Table 26.

Company ↓	Bids				Penalty
	Route 1	Route 2	Route 3	Route 4	↓
1	4	5	-	-	1
2	-	4	-	4	0
3	3	-	$\boxed{2}$	-	1
4	-	-	4	5	1
Penalty	1	1	2	1	

Table 25

Step-II. We now select row 1 because it has the highest penalty rating 1.we look at the cell with the least cost in the row 1 is to be identified. Here it is the cell (1, 1) with entry 4. So we assign 4 to (1, 1). Hence row and column are deleted corresponding to cell (1, 1) and resulting data shown in Table 27.

Company ↓	Bids			Penalty
	Route 1	Route 2	Route 4	↓
1	$\boxed{4}$	5	-	1
2	-	4	4	0
4	-	-	5	-
Penalty	-	1	1	

Table 26

Step-III. We now select column 2 because it has the highest penalty rating 1.we look at the cell with the least cost in the column 2 is to be identified. Here it is the cell (4, 4) with entry 4. but we cannot assign this cost, in this case we assign 4 to (2, 2) and 5 to (4, 4).

Company ↓	Bids		Penalty ↓
	Route 2	Route 4	
2	$\boxed{4}$	4	0
4	-	$\boxed{5}$	-
Penalty	-	1	

Table 27

Step-IV. The whole procedure can be done in a single table as given below

Company ↓	Bids			
	Route 1	Route 2	Route 3	Route 4
1	$\boxed{4}$	5	-	-
2	-	$\boxed{4}$	-	4
3	3	-	$\boxed{2}$	-
4	-	-	4	$\boxed{5}$

Table 28

Hence optimal solution is: 1→1, 2→2, 3→3, 4→4
The minimum cost = 4+4+2+5= 15 Rs.

Conclusion

In this paper, a new technique namely least cost assignment technique is introduced to solve assignment problem. This method is applicable for all kind of assignment problems, whether maximize or minimize objective function. In this method we do not require to mark any row or column to draw minimum number of lines, so this technique is systematic, easy to apply and consume less time in comparison to another techniques. This technique gives better optimal solution in comparison with Hungarian method.

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