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Generalized Degenerated Bernoulli Numbers and Polynomials

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Abstract: The Generalized degenerate Bernoulli numbers $B\,(\lambda)$ can be defined by means of the exponential generating function $(t)^{\alpha} / \left[(1+\lambda t)^{\frac{1}{\lambda}} - 1 \right]^{\alpha}$. As further applications we derive several identities,recurrences,andcongruences involving theGeneralized Bernoulli numbers, Generalizeddegenerate Bernoulli numbers and polynomials.

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1. Introduction

Carlitz[1] defined the generalized degenerate Bernoulli numbers $B\,(\lambda)$ by means of the generating function

(1.1) $$\frac{(t)^{\alpha}}{\left[(1+\lambda t)^{\frac{1}{\lambda}}-1\right]^{\alpha}}=\sum_{m=0}^{\infty} B_{m}(\lambda)\frac{t^{m}}{m!}$$

we have, $B_{m}(0) =$ B_{m}, the ordinary generalized Bernoulli number In[2] Carlitz proved many properties of $B\,(\lambda)$. He also pointed out that $B\,(\lambda)$ is a polynomials in λ with degree ≤ *m*, we have

$$B\ (\lambda)=1$$

$$B\ (\lambda)=\frac{-\alpha}{2}+\frac{\alpha\lambda}{2} \quad \text{and so on.}$$

Carlitz[2] also defined the generalized degenerate Bernoulli polynomials $B_{m}(\lambda,x)$ for $\lambda \neq 0$ by means of the generating function.

(1.2) $$\frac{(t)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}}}{\left[(1+\lambda t)^{\frac{1}{\lambda}}-1\right]^{\alpha}}=\sum_{m=0}^{\infty} B_{m}(\lambda,x)\frac{t^{m}}{m!}$$

where λμ=1. These are polynomials in λ and *x* with rational coefficients. We often write $B\ (\lambda)$ *for* $B\ (\lambda,0)$, and refer to the polynomial $B_{m}(\lambda)$ as ageneralized degenerate Bernoulli number. The first few are

$$B\ (\lambda,x)=1$$

$$B\ (\lambda,x)=x-\frac{\alpha}{2}+\frac{\alpha}{2}\lambda \quad \text{and so on}$$

Clearly, we have

$$B_{m}(\lambda,x)= B_{m}(\lambda,x)$$

2. A Recurrence Relation of $B_{m}^{\alpha}(\lambda,x)$

In this section, we derive the following recurrence relation for $B_{m}(\lambda,x)$

(2.1) $$B(\lambda, x) = \sum_{k=0}^{m} \binom{m}{k} B(\lambda) \left(\frac{x}{\lambda}\right)_{m-k}$$

Proof: We know that the generating function of generalized degenerate Bernoulli polynomial

$$\frac{(t)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}}}{\left[(1+\lambda t)^{\frac{1}{\lambda}} - 1\right]^{\alpha}} = \sum_{m=0}^{\infty} B(\lambda, x)\frac{t^m}{m!}$$

By (1.1) we get

$$\sum_{m=0}^{\infty} B_m(\lambda)\frac{t^m}{m!}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{m=0}^{\infty} B_m(\lambda, x)\frac{t^m}{m!}$$

By the Binomial expansion

$$[(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + n(n-1)(n-2)\frac{x^3}{3!} + \ldots]$$

$$\sum_{m=0}^{\infty} B_m(\lambda)\frac{t^m}{m!}\sum_{m=0}^{\infty}\left(\frac{x}{\lambda}\right)_m \frac{t^m}{m!} = \sum_{m=0}^{\infty} B_m(\lambda, x)\frac{t^m}{m!}$$

By the Cauchy product rule

$$B_m(\lambda, x) = \sum_{k=0}^{m}\binom{m}{k} B(\lambda)\left(\frac{x}{\lambda}\right)_{m-k}$$

where $\left(\frac{x}{\lambda}\right)_m = [x(x-\lambda)(x-2\lambda).....(x-(m-1)\lambda]$

Particular case: It is interesting to note that (2.1) reduces to the well-known recurrence relation of degenerate Bernoulli polynomial for $\alpha = 1$.

(2.2) $$B_m(\lambda, x) = \sum_{k=0}^{m}\binom{m}{k} B(\lambda)\left(\frac{x}{\lambda}\right)_{m-k}$$

3. Properties of generalized Degenerate Bernoulli polynomial

In this section, some of well-known properties ofgeneralized Degenerate Bernoulli polynomials are derived from the generating function (1.2)

Property-1

(3.1) $$\mathcal{B}_m(\lambda, x+y) = \sum_{k=0}^{m} \binom{m}{k} \mathcal{B}(\lambda, x) \left(\frac{y}{\lambda}\right)_{m-k}$$

Proof: Now put $x \to x+y$ in (1.2)

$$\frac{(t)^{\alpha}(1+\lambda t)^{\mu(x+y)}}{\left[(1+\lambda t)^{\frac{1}{\lambda}}-1\right]^{\alpha}} = \sum_{m=0}^{\infty} \mathcal{B}_m(\lambda, x+y)\frac{t^m}{m!}$$

$$\frac{(t)^{\alpha}(1+\lambda t)^{\mu x}(1+\lambda t)^{\mu y}}{\left[(1+\lambda t)^{\frac{1}{\lambda}}-1\right]^{\alpha}} = \sum_{m=0}^{\infty} \mathcal{B}(\lambda, x+y)\frac{t^m}{m!}$$

By the equation (1.2)

$$\sum_{m=0}^{\infty} \mathcal{B}_m(\lambda, x)\frac{t^m}{m!}(1+\lambda t)^{\mu y} = \sum_{m=0}^{\infty} \mathcal{B}_m(\lambda, x+y)\frac{t^m}{m!}$$

By the help of Binomial expansion

$$(1+\lambda t)^{\mu y} = \sum_{m=0}^{\infty} \left(\frac{y}{\lambda}\right)_m \frac{t^m}{m!}$$

Therefore

$$\sum_{m=0}^{\infty} \mathcal{B}_m(\lambda, x)\frac{t^m}{m!}\sum_{m=0}^{\infty} \left(\frac{y}{\lambda}\right)_m \frac{t^m}{m!} = \sum_{m=0}^{\infty} \mathcal{B}_m(\lambda, x+y)\frac{t^m}{m!}$$

By the Cauchy product rule

$$\left(\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}\right) . \left(\sum_{n=0}^{\infty} b_n \frac{t^n}{n!}\right) = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}$$

where

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k . b_{n-k}$$

$$\mathcal{B}_m(\lambda, x+y) = \sum_{k=0}^{m} \binom{m}{k} \mathcal{B}(\lambda, x) \left(\frac{y}{\lambda}\right)_{m-k}$$

Particular case: When $\alpha = 1$, we get the ordinary Degenerate Bernoulli polynomial

(3.2)
$$\mathcal{B}_m(\lambda, x+y) = \sum_{k=0}^{m} \binom{m}{k} \mathcal{B}(\lambda, x) \left(\frac{y}{\lambda}\right)_{m-k}$$

Here $y=1$ then,

(3.3)
$$\mathcal{B}_m(\lambda, x+1) = \sum_{k=0}^{m} \binom{m}{k} \mathcal{B}(\lambda, x) \left(\frac{1}{\lambda}\right)_{m-k}$$

Property-2

(3.4)
$$\frac{d}{dx} \mathcal{B}_m^{\alpha}(\lambda, x) = \lambda^{-1} \mathcal{B}_m(\lambda, x)$$

Proof:-By the generating function of generalized degenerate Bernoulli polynomials

$$\frac{(t)^{\alpha}(1+\lambda t)^{\mu x}}{\left[(1+\lambda t)^{\mu} - 1\right]^{\alpha}} = \sum_{m=0}^{\infty} \mathcal{B}_m(\lambda, x) \frac{t^m}{m!}$$

Differentiate above equation with respect to x

$$\frac{\mu(t)^{\alpha}(1+\lambda t)^{\mu x}}{\left[(1+\lambda t)^{\mu}-1\right]^{\alpha}}=\sum_{m=0}^{\infty}\frac{d}{dx}\ \mathcal{B}_{m}(\lambda,x)\frac{t^{m}}{m!}$$

$$\mu.\sum_{m=0}^{\infty}\ \mathcal{B}_{m}(\lambda,x)\frac{t^{m}}{m!}=\sum_{m=0}^{\infty}\frac{d}{dx}\ \mathcal{B}_{m}(\lambda,x)\frac{t^{m}}{m!}$$

Equating the coefficients

$$\frac{d}{dx}\ \mathcal{B}_{m}(\lambda,x)=\mu\ \mathcal{B}_{m}(\lambda,x)\ \text{Where } \mu\lambda=1$$

$$\frac{d}{dx}\ \mathcal{B}_{m}(\lambda,x)=\lambda^{-1}\ \mathcal{B}_{m}(\lambda,x)$$

Property-3

(3.5) $$\mathcal{B}^{\alpha}_{m}(\lambda,\alpha-x)=(-1)^{m}\ \mathcal{B}_{m}(\lambda,x)$$

Proof: By equation (1.2)

$$\frac{(t)^{\alpha}(1+\lambda t)^{\mu x}}{\left[(1+\lambda t)^{\mu}-1\right]^{\alpha}}=\sum_{m=0}^{\infty}\ \mathcal{B}^{\alpha}_{m}(\lambda,x)\frac{t^{m}}{m!}$$

Replace $x\rightarrow\alpha-x$ in the above equation

$$\frac{(t)^{\alpha}(1+\lambda t)^{\mu(\alpha-x)}}{\left[(1+\lambda t)^{\mu}-1\right]^{\alpha}}=\sum_{m=0}^{\infty}\ \mathcal{B}_{m}(\lambda,\alpha-x)\frac{t^{m}}{m!}$$

$$\frac{(t)^{\alpha}(1+\lambda t)^{\mu\alpha}(1+\lambda t)^{-\mu x}}{\left[(1+\lambda t)^{\mu}-1\right]^{\alpha}}=\sum_{m=0}^{\infty} \mathcal{B}_m(\lambda,\alpha-x)\frac{t^m}{m!}$$

$$\frac{(t)^{\alpha}(1+\lambda t)^{\mu\alpha}(1+\lambda t)^{-\mu x}}{(1+\lambda t)^{\mu\alpha}\left[1-(1+\lambda t)^{-\mu}\right]^{\alpha}}=\sum_{m=0}^{\infty} \mathcal{B}_m(\lambda,\alpha-x)\frac{t^m}{m!}$$

$$\frac{(t)^{\alpha}(1+\lambda t)^{-\mu x}}{\left[1-(1+\lambda t)^{-\mu}\right]^{\alpha}}=\sum_{m=0}^{\infty} \mathcal{B}_m(\lambda,\alpha-x)\frac{t^m}{m!}$$

$$-\frac{(t)^{\alpha}(1+\lambda t)^{-\mu x}}{\left[(1+\lambda t)^{-\mu}-1\right]^{\alpha}}=\sum_{m=0}^{\infty} \mathcal{B}_m(\lambda,\alpha-x)\frac{t^m}{m!}$$

$$\sum_{m=0}^{\infty} \mathcal{B}\,(\lambda,x)\frac{(-t)^m}{m!}=\sum_{m=0}^{\infty} \mathcal{B}\,(\lambda,\alpha-x)\frac{t^m}{m!}$$

$$\sum_{m=0}^{\infty} \mathcal{B}\,(\lambda,x)\frac{(-t)^m}{m!}=\sum_{m=0}^{\infty} \mathcal{B}\,(\lambda,\alpha-x)\frac{t^m}{m!}$$

Equating the coefficients

$$\mathcal{B}_m(\lambda\,,\alpha-x)=(-1\ )^m\ \mathcal{B}_m(\lambda,x)$$

Property-4

(3.6) $$B_m^{\alpha}(\lambda,x+1)-B_m^{\alpha}(\lambda,x)=m\,\mathcal{B}_{m-1}^{-1}(\lambda,x)$$

Proof: By equation (1.2)

$$\frac{(t)^{\alpha}(1+\lambda t)^{\mu x}}{\left[(1+\lambda t)^{\mu}-1\right]^{\alpha}}=\sum_{m=0}^{\infty} B_{m}(\lambda,x)\frac{t^{m}}{m!} \tag{3.7}$$

Replace $x \to x+1$ in the above equation.

$$\frac{(t)^{\alpha}(1+\lambda t)^{\mu(x+1)}}{\left[(1+\lambda t)^{\mu}-1\right]^{\alpha}}=\sum_{m=0}^{\infty} B\,(\lambda,x+1)\frac{t^{m}}{m!} \tag{3.8}$$

Subtracting (3.8)-(3.7)

$$\frac{(t)^{\alpha}(1+\lambda t)^{\mu(x+1)}}{\left[(1+\lambda t)^{\mu}-1\right]^{\alpha}}-\frac{(t)^{\alpha}(1+\lambda t)^{\mu x}}{\left[(1+\lambda t)^{\mu}-1\right]^{\alpha}}=\sum_{m=0}^{\infty} B\,(\lambda,x+1)\frac{t^{m}}{m!}-\sum_{m=0}^{\infty} B\,(\lambda,x)\frac{t^{m}}{m!}$$

$$\frac{(t)^{\alpha}(1+\lambda t)^{\mu x}\left[(1+\lambda t)^{\mu}-1\right]}{\left[(1+\lambda t)^{\mu}-1\right]^{\alpha}}=\sum_{m=0}^{\infty}\left[B_{m}^{\alpha}(\lambda,x+1)-B_{m}^{\alpha}(\lambda,x)\right]\frac{t^{m}}{m!}$$

$$\frac{(t)^{\alpha-1+1}(1+\lambda t)^{\mu x}\left[(1+\lambda t)^{\mu}-1\right]}{\left[(1+\lambda t)^{\mu}-1\right]^{\alpha-1+1}}=\sum_{m=0}^{\infty}\left[B_{m}^{\alpha}(\lambda,x+1)-B_{m}^{\alpha}(\lambda,x)\right]\frac{t^{m}}{m!}$$

$$\frac{t.(t)^{\alpha-1}(1+\lambda t)^{\mu x}}{\left[(1+\lambda t)^{\mu}-1\right]^{\alpha-1}}=\sum_{m=0}^{\infty}\left[B_{m}^{\alpha}(\lambda,x+1)-B_{m}^{\alpha}(\lambda,x)\right]\frac{t^{m}}{m!}$$

$$t\sum_{m=0}^{\infty} B_{m}^{\alpha-1}(\lambda,x)\frac{t^{m}}{m!}=\sum_{m=0}^{\infty}\left[B_{m}^{\alpha}(\lambda,x+1)-B_{m}^{\alpha}(\lambda,x)\right]\frac{t^{m}}{m!}$$

$$\sum_{m=0}^{\infty} B_m^{\alpha-1}(\lambda,x)\frac{t^{m+1}}{m!}=\sum_{m=0}^{\infty}\left[B_m^{\alpha}(\lambda,x+1)-B_m^{\alpha}(\lambda,x)\right]\frac{t^m}{m!}$$

Equating the coefficient of $\frac{t^m}{\text{m!}}$

$$B_m^{\alpha}(\lambda,x+1)-B_m^{\alpha}(\lambda,x)=m\,B_{m-1}^{\alpha-1}(\lambda,x)$$

Property-4

(3.9) $$B_m^{-1}(\lambda,x)=\frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B\ (\lambda,x)\left(\frac{1}{\lambda}\right)_{m+1-k}$$

Proof:-By equation (3.6)

$$B_m(\lambda,x+1)=\sum_{k=0}^{m}\binom{m}{k}B\ (\lambda,x)\left(\frac{1}{\lambda}\right)_{m-k}$$

$$B_m(\lambda,x+1)=\sum_{k=0}^{m-1}\binom{m}{k}B\ (\lambda,x)\left(\frac{1}{\lambda}\right)_{m-k}+\binom{m}{m}B\ (\lambda,x)\left(\frac{1}{\lambda}\right)_{m-m}$$

$$B_m(\lambda,x+1)-B_m(\lambda,x)=\sum_{k=0}^{m-1}\binom{m}{k}B\ (\lambda,x)\left(\frac{1}{\lambda}\right)_{m-k}$$

But we know that by (3.5)

$$B_m^{\alpha}(\lambda,x+1)-B_m^{\alpha}(\lambda,x)=m\,B_{m-1}^{-1}(\lambda,x)$$

(3.10) $$m\,B_{m-1}^{-1}(\lambda,x)=\sum_{k=0}^{m-1}\binom{m}{k}\ B\ (\lambda,x)\left(\frac{1}{\lambda}\right)_{m-k}$$

Now put $m \to m+1$ in the above equation

$$\mathcal{B}^{-1}(\lambda, x) = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}\mathcal{B}\ (\lambda, x)\left(\frac{1}{\lambda}\right)_{m+1-k} \tag{3.11}$$

Now put $x=0$ in equation (3.10), then

$$m\mathcal{B}_{-1}^{-1}(\lambda,0) = \sum_{k=0}^{m-1}\binom{m}{k}\mathcal{B}\ (\lambda,0)\left(\frac{1}{\lambda}\right)_{m-k}$$

By definition

$$\mathcal{B}_m(\lambda,0) = \mathcal{B}_m(\lambda)$$

Therefore

$$m\ \mathcal{B}_{m-1}^{-1}(\lambda) = \sum_{k=0}^{m-1}\binom{m}{k}\ \mathcal{B}(\lambda)\left(\frac{1}{\lambda}\right)_{m-k}. \tag{3.12}$$

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