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Some Fixed Point Theorems of Generalized Contractive Mappings in Complete Cone Metric Spaces

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Abstract: The purpose of this paper is to establish some fixed point theorems for generalized contractive mappings in the framework of complete cone metric spaces. The results presented in this paper synthesize and generalize the corresponding results of Huang and Zhang [9], Rezapour and Hamlbarani[16] and many others from the current literature.

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1. Introduction and Preliminaries

The Well-known Banach contraction principle and its several generalization in the setting of metric spaces play a central role for solving many problems of nonlinear analysis. For example, see[2, 6, 7, 12, 13].

Recently, Huang and Zhang[9] used the notion of cone metric spaces as a generalization of metric spaces. They have replaced the real numbers (as the co-domain of a "metric") by an ordered Banach space. The authors described the convergence in cone metric spaces and introduced their completeness. Then they proved some fixed point theorems for contractive single-valued mappings in such spaces. In their theorems cone is normal. For more fixed point results in cone metric spaces, see[1, 3, 10, 15, 16, 18]. In this paper we establish some fixed point theorems for generalized contraction mappings in the framework of cone metric spaces.

Let E be a real Banach space. A subset P of E is called a cone whenever the following conditions hold:

(c_1) P is closed, nonempty and $P \neq \{0\}$;

(c_2) $a,b \in R; a,b \geq 0$ and $x,y \in P$ imply $ax+by \in P$;
(c_3) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y-x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y-x \in \operatorname{int} P$ (interior of P). If $\operatorname{int} P \neq \phi$ then P is called a solid cone (see[17]).

There exist two kinds of cones-normal (with the normal constant K) and non-normal ones[7].

Let E be a real Banach space, $P \subset E$ a cone and $\leq$ partial ordering defind by P. Then P is called normal if there is a number K > 0 such that for all $x, y \in P$,

(1.1) $\quad 0 \leq x \leq y$ imply $\|x\| \leq K\|y\|$

or equivalently, if $(\forall n)$ $x_n \leq y_n \leq z_n$ and

(1.2) $\quad \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x$ imply $\lim_{n\to\infty} y_n = x$

The least positive number K satisfying (1.1) is called the normal constant of P. It is clear that K $\geq$ 1.

Example1.1. [7]Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ on $P = \{x \in E : x(t) \geq 0\}$. This cone is not normal. Consider, for example, $x_n(t) = \dfrac{t^n}{n}$ and $y_n(t) = \dfrac{1}{n}$. Then $0 \leq x_n \leq y_n$, and $\lim_{n\to\infty} y_n = 0$, but $\|x_n\| = \max_{t\in[0,1]} \left|\dfrac{t^n}{n}\right| + \max_{t\in[0,1]} \left|t^{n-1}\right| = \dfrac{1}{n} + 1 > 1$;
hence x_n does not coverge to zero.
It follows by (1.2) that P is a non-normal cone.

Definition 1.1. [9, 19]Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

(d_1) $0 \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;
(d_2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(d3) $d(x,y) \le d(x,z) + d(z,y)$ for all $x,y,z \in X$.
Then d is called a cone metric[9] or K-metric[19] on X and (X,d) is called a cone metric[9] or K-metric space[19] (we shall use the first term).

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

Example 1.2. [9]Let $E = \mathbb{R}^2$, $P = \{(x,y) \in \mathbb{R}^2 : x \ge 0, y \ge 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ defined by $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \ge 0$ is a constant. Then (X,d) is a cone metric space with normal cone P where K=1.

Example 1.3. [15] Let $E = \ell^2$, $P = \{\{x_n\}_{n\ge1} \in E : x_n \ge 0, \forall n\}$, (X,ρ) is a metric space, and $d : X \times X \to E$ defined by $d(x,y) = \left\{\dfrac{\rho(x,y)}{2^n}\right\}_{n\ge1}$. Then (X,d) is a cone metric space.

Clearly, the above examples show that the class of cone metric spaces contains the class of metric spaces.

Definition 1.2. [9]Let (X,d) be a cone metric space. We say that $\{x_n\}$ is:
(i) a Cauchy sequence if for every ε in E with $0 \ll \varepsilon$, then there is an N such that for all $n,m > N$, $d(x_n, x_m) \ll \varepsilon$;
(ii) a convergent sequence if for every ε in E with $0 \ll \varepsilon$, then there is an N such that for all $n > N$, $d(x_n, x) \ll \varepsilon$ for some fixed x in X.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Let us recall[9] that if P is a normal solid cone, then $x_n \in X$ is a Cauchy sequence if and only if $\|d(x_n, x_m)\| \to 0$ as $n,m \to \infty$.
Further, $x_n \in X$ converges to $x \in X$ if and only if $\|d(x_n, x)\| \to 0$ as $n \to \infty$.

In the sequel we assume that E is a real Banach space and that P is a normal solid cone in E, that is, normal cone with $\operatorname{int} P \ne \phi$. The last assumption is necessary in order to obtain reasonable results connected with convergence and continuity. The partial ordering induced by the cone P will be denoted by $\leq$.

Generalized Contraction Mapping

Let X be a cone metric space and $T: X \to X$ be a mapping. Then T is called generalized contractive mapping if it satisfies the following condition:

$$(1.3)\quad d(Tx,Ty) \leq ad(x,y) + b[d(x,Tx) + d(y,Ty)] + c[d(x,Ty) + d(y,Tx)]$$

for all $x, y \in X$ and $a,b,c \in [0,1)$ are constants such that $a + 2b + 2c < 1$.

Remark 1.1.

(1) If $b = c = 0$ and $a \in [0,1)$, then (1.3) reduces to contraction mapping defined by Banach[4].
(2) If $a = c = 0$ and $b \in [0,1/2]$, then (1.3) reduces to contraction mapping defined by Kannan[11].
(3) If $c = 0$ and $a,b \in [0,1/2]$, then (1.3) reduces to contraction mapping defined by Fisher[8].
(4) If $a = b = 0$ and $c \in [0,1/2]$, then (1.3) reduces to contraction mapping defined by Chaterjee[5].
(5) If $b = 0$ and $a,c \in [0,1)$, then (1.3) reduces to contraction mapping defined by Reich[14].

2. Main Results

In this section we shall prove some fixed point theorems for generalized contractive mappings.

Theorem 2.1. *Let (X,d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T: X \to X$ satisfies generalized contractive condition (1.3) with $a + 2b + 2c < 1$ where $a,b,c \in [0,1)$ are constants. Then T has a unique fixed point in X and for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.*

Proof. Choose $x_0 \in X$. Set

$x_1 = Tx_0, x_2 = Tx_1 = T^2 x_0, \ldots, x_{n+1} = Tx_n = T^{n+1} x_0, \ldots,$ we have

$$
\begin{aligned}
d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\
&\le ad(x_n, x_{n-1}) + b[d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})] \\
&\quad + c[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)] \\
&= ad(x_n, x_{n-1}) + b[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\
&\qquad + c[d(x_n, x_n) + d(x_{n-1}, x_{n+1})] \\
&\le ad(x_n, x_{n-1}) + b[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\
&\qquad + c[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
&\le (a+b+c)d(x_n, x_{n-1}) + (b+c)d(x_n, x_{n+1}).
\end{aligned}
$$

That is,

$$(1-b-c)d(x_{n+1}, x_n) \le (a+b+c)d(x_n, x_{n-1})$$

which implies that

$$d(x_{n+1}, x_n) \le hd(x_n, x_{n-1})$$

where

$$h = \frac{a+b+c}{1-b-c}.$$

As $a+2b+2c<1$, we obtain that $h<1$.

Now

$$d(x_{n+1}, x_n) \le hd(x_n, x_{n-1}) \le h^2 d(x_{n-1}, x_{n-2}) \le \ldots \le h^n d(x_1, x_0).$$

So for $n>m$, we have

$$
\begin{aligned}
d(x_n, x_m) &\le d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \ldots + d(x_{m+1}, x_m) \\
&\le (h^{n-1} + h^{n-2} + \ldots + h^m)d(x_1, x_0) = \frac{h^m}{1-h} d(x_1, x_0).
\end{aligned}
$$

We get $\qquad \|d(x_n, x_m)\| \le \frac{h^m}{1-h} K \|d(x_1, x_0)\|.$

This implies $\qquad \|d(x_n, x_m)\| \to 0$ as $n, m \to \infty$.

Hence $\{x_n\}$ is a Cauchy sequence.

By the completeness of X, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$.

Since

$$d(Tu,u) \le d(Tx_n,Tu) + d(Tx_n,u)$$

$$\le \left(\frac{a+b+c}{1-b-c}\right) d(x_n,u) + \left(\frac{1+b+c}{1-b-c}\right) d(x_{n+1},u).$$

As $x_n \to u$, $x_{n+1} \to u$ $(n \to \infty)$, we get

$$\|d(Tu,u)\| \le K\left(\left(\frac{a+b+c}{1-b-c}\right)\|d(x_n,u)\| + \left(\frac{1+b+c}{1-b-c}\right)\|d(x_{n+1},u)\|\right) \to 0.$$

Hence $\|d(Tu,u)\| = 0$.
This implies $Tu = u$. So u is a fixed point of T.

Now if v is another fixed point of T, then

$$d(u,v) = d(Tu,Tv) \le (a+2c)d(u,v).$$

Since $(a+2c) < 1$, hence $\|d(u,v)\| = 0$ and $u = v$.
Therefore the fixed point of T is unique.

Theorem 2.2. *Let (X,d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T: X \to X$ satisfies for some positive integer n:*

$$d(T^n x, T^n y) \le a_n d(x,y) + b_n[d(x,T^n x) + d(y,T^n y)] + c_n[d(x,T^n y) + d(y,T^n x)]$$

for all $x, y \in X$ and $a_n, b_n, c_n \in [0,1)$ are constants such that $a_n + 2b_n + 2c_n < 1$. Then T has a unique fixed point in X.

Proof. From Theorem 2.1, T^n has a unique fixed point u. But $T^n(Tu) = T(T^n u) = Tu$, so Tu is also a fixed point of T^n. Hence $Tu = u$, that is, u is a fixed point of T. Since the fixed point of T is also a fixed point of T^n, the fixed point of T is unique.
Huang and Zhang[9] and Rezapour and Hamlbarani[16] proved following various form of Banach contraction principle in a normal cone metric space and in a cone metric space respectively.

Theorem 1[9] and Theorem 2.3[16]. *Let* (X,d) *be a complete cone metric space. Suppose the mapping* $T: X \to X$ *satisfies the contractive condition*

$$d(Tx,Ty) \le kd(x,y),$$

for all $x, y \in X$ *, where* $k \in [0,1)$ *is a constant. Then T has a unique fixed point in X and for any* $x \in X$ *, the iterative sequence* $\{T^n x\}$ *converges to the fixed point.*

Theorem 3[9] and Theorem 2.6[16]. *Let* (X,d) *be a complete cone metric space. Suppose the mapping* $T: X \to X$ *satisfies the contractive condition*

$$d(Tx,Ty) \le k[d(x,Tx) + d(y,Ty)],$$

for all $x, y \in X$ *, where* $k \in [0,1/2)$ *is a constant. Then T has a unique fixed point in X and for any* $x \in X$ *, the iterative sequence* $\{T^n x\}$ *converges to the fixed point.*

Theorem 4[9] and Theorem 2.7[16]. *Let* (X,d) *be a complete cone metric space. Suppose the mapping* $T: X \to X$ *satisfies the contractive condition*

$$d(Tx,Ty) \le k[d(x,Ty) + d(y,Tx)],$$

for all $x, y \in X$ *, where* $k \in [0,1/2)$ *is a constant. Then T has a unique fixed point in X and for any* $x \in X$ *, the iterative sequence* $\{T^n x\}$ *converges to the fixed point.*

Remark 2.1. Above Theorems[9, 16] follow from Theorem 2.1 of this paper by taking

(i) $b = c = 0$ and $a = k$,
(ii) $a = c = 0$ and $b = k$,
(iii) $a = b = 0$ and $c = k$,
respectively in it.

Corollary 2[9]. *Let* (X,d) *be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping* $T: X \to X$ *satisfies for some positive integer n*

$$d(T^n x, T^n y) \le kd(x,y),$$

for all $x, y \in X$ *, where* $k \in [0,1)$ *is a constant. Then T has a unique fixed point in X.*

Remark 2.2. Above Corollary of[9] follows from Theorem 2.2 of this paper by taking $b_n = c_n = 0$ and $a_n = k$ in it.

Precisely, Theorem 2.1 synthesizes and generalizes all the results[9, 16].
We conclude with an example.

Let $E = \mathbb{R}^2$, the Euclidean plane, and $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ a normal cone in P. Let $X = \{(x,0) \in \mathbb{R}^2 : 0 \leq x \leq 1\} \cup \{(0,x) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$. The mapping $d : X \times X \to E$ is defined by

$$d\big((x,0),(y,0)\big) = \left(\frac{5}{3}|x-y|, |x-y|\right),$$

$$d\big((0,x),(0,y)\big) = \left(|x-y|, \frac{2}{3}|x-y|\right),$$

$$d\big((x,0),(0,y)\big) = d\big((0,y),(x,0)\big) = \left(\frac{5}{3}x + y, x + \frac{2}{3}y\right).$$

Then (X, d) is a complete cone metric space.

Let mapping $T : X \to X$ with

$$T\big((x,0)\big) = (0,x) \text{ and } T\big((0,x)\big) = \left(\frac{1}{2}x, 0\right).$$

Then T satisfies the generalized contractive condition

$$\begin{aligned} d\big(T((x,x')),T((y,y'))\big) \leq\ & ad\big((x,x'),(y,y')\big) \\ & + b[d\big((x,x'),T((x,x'))\big) + d\big((y,y'),T((y,y'))\big)] \\ & + c[d\big((x,x'),T((y,y'))\big) + d\big((y,y'),T((x,x'))\big)] \end{aligned}$$

for all $(x,x'),(y,y')\in X$ with the constant $\lambda = a+2b+2c<1$, where a,b,c are such that $a=b=c=\frac{1}{6}$. Then it is obvious that T has a unique fixed point $(0,0)\in X$, where $\lambda=\frac{5}{6}\in[0,1)$.

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