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Space-time Admitting W_4-Curvature Tensor

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Abstract: In this paper, we have studied space-time with W_4-curvature tensor and proved that a 4-dimensional relativistic W_4-flat space-time satisfying Einstein's field equation with cosmological constant, the energy-momentum tensor is covariant constant. It is also observed that in a 4-dimensional relativistic space-time M has conservative W_4-curvature tensor if and only if the energy momentum tensor is Codazzi tensor provided that the scalar curvature is constant in both the cases.

Keywords: W_4-curvature tensor; Conservative W_4-curvature tensor; Einstein's field equation; Perfect fluid space-time; Energy-momentum tensor; Generalized Robertson-Walker space-time.

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1. Introduction

The aim of the present work is to study certain investigations in general theory of relativity and cosmology by the coordinate free method of differential geometry. The basic difference between Riemannian and semi-Riemannian geometry are (i) the existence of null vector (i.e. $g(v,v)=0$, for $v \neq 0$ where g is the metric tensor) in semi-Riemannian manifold but not Riemannian manifold, (ii) the signature of metric tensor g in semi-Riemannian manifold is $(-,-,...-,+,+,...,+)$ but in a Riemannian manifold the signature of g is $(+,+,...,+)$. Lorentzian manifold is a spacial case of semi-Riemannian manifold. The signature of metric tensor g in Lorentzian manifold is $(-,+,+,...,+)$. A Lorentzian manifold consists of three types of vectors such as time-like (i.e. $g(v,v)<0$), spacelike (i.e. $g(v,v)>0$) and null vector (i.e. $g(v,v)=0$, for $v\neq 0$). In general, a Lorentzian manifold (M,g) may not have a globally time-like vector field. If (M,g) admits a globally time-like vector field, it is called time orientable Lorentzian manifold, physically known as space-time. The foundation of general relativity is based on a 4-dimensional space-time manifold which is the stage of present modeling of the physical world a torsionless, time-oriented Lorentzian manifold (M,g).

An n-dimensional generalized Robertson-Walker[1-3] (GRW) space-time with $n\geq 3$ is a Lorentzian manifold which is a warped product of an open interval I of $\mathbb{R}$ and a $(n-1)-$ dimensional Riemannian manifold. These Lorentzian manifolds broadly extend the classical Robertson-Walker (RW) space-time. RW space-time is regarded as cosmological models since it is spatially homogenous and spatially isotropic whereas GRW space-time serves as inhomogeneous extension of RW space-times that admit an isotropic radiation.

In general relativity the matter content of the space-time is described by the energy momentum tensor. The matter content is assumed to be a fluid having density and pressure and possessing dynamical and kinematical quantities like velocity, acceleration, vorticity, shear and expansion. In a perfect fluid space-time[4], the energy momentum tensor T of type $(0,2)$ is of the form

(1.1) $$T(X,Y)=(\sigma+\rho)A(X)A(Y)+\rho g(X,Y),$$

where ρ is the isotropic pressure, σ is the energy density and A is a non-zero one-form such that $g(X,\mu)=A(X)$ for all X,μ and μ is the velocity vector field such that $g(\mu,\mu)=-1$. The fluid is called perfect because of the absence of heat conduction terms and stress terms corresponding to viscosity. Perfect- fluid space-times in a language of differential geometry are called quasi-Einstein spaces where A is metrically equivalent to a unit space-like vector field. If the isotropic pressure ρ is vanish in perfect fluid then it is said to be dust fluid. In a dust fluid space-time[4], the energy momentum tensor T of type $(0,2)$ is of the form

(1.2) $$T(X,Y)=\sigma A(X)A(Y),$$

The Einstein's field equation[4] with cosmological constant is given by

(1.3) $$S(X,Y)-\frac{r}{2}g(X,Y)+\lambda\, g(X,Y)=kT(X,Y),$$

where S and r denotes the Ricci tensor and scalar curvature respectively, λ is the cosmological constant, T is the energy momentum tensor and $k \neq 0$ .Einstein's field equation[4] without cosmological constant is given by

(1.4) $$S(X,Y)-\frac{r}{2}g(X,Y)=kT(X,Y).$$

The Einstein's field equations (1.3) and (1.4) imply that the energy-momentum tensor is conservative. This requirement is satisfied if the energy-momentum tensor is covariant constant. M. C. Chaki and Sarbari Ray[5] showed that a general relativistic space-time with covariant constant energy-momentum tensor is Ricci symmetric, that is, $\nabla S = 0,$ where S is Ricci tensor of the space-time.

A symmetric $(0,2)$ type tensor field E on a semi-Riemannian manifold (M^n, g) is said to be a Codazzi tensor if it satisfies the Codazzi equation

(1.5) $$(\nabla_U E)(V,X)=(\nabla_V E)(U,X),$$

for arbitrary vector fields U ,V and X . The geometrical and topological consequences of the existence of a non-trivial Codazzi tensor on a Riemannian manifold have been studied by Derdzinski and Shen[4].

Pokhariyal and Mishra[6] introduced some curvature tensors similar to projective curvature tensor on an n-dimensional Riemannian manifold whose physical significance and geometric properties have been studied by several authors. Our work is confined to study geometric behavior of W_4 – curvature tensor on relativistic space-time. The W_4 – curvature tensor on an n-dimensional Riemannian manifold is defined as

(1.6) $$W_4(U,V,X,Y)=R(U,V,X,Y)+\frac{1}{n-1}[g(U,X)S(V,Y)-g(U,V)S(X,Y)],$$

where $R(U,V,X,Y)=g\left(R(U,V)X,Y\right)$ is the Riemannian curvature.

2. W_4 – Flat Space-Time

Let M be a 4-dimensional space-time of general relativity, then in view of equation (1.6), we have

(2.1) $$W_4(U,V,X,Y) = R(U,V,X,Y) + \frac{1}{3}[g(U,X)S(V,Y) - g(U,V)S(X,Y)].$$

If $W_4(U,V,X,Y) = 0$, then from equation (2.1), we have

(2.2) $$R(U,V,X,Y) = \frac{1}{3}[g(U,V)S(X,Y) - g(U,X)S(V,Y)].$$

Putting $V = Y = e_i$ and taking summation over i, $1 \le i \le 4$, we obtain

(2.3) $$S(U,X) = \frac{-r}{2} g(U,X).$$

Thus, a W_4 – curvature flat space-time is an Einstein manifold. Next, using equation (2.3) in equation (2.2), we get

(2.4) $$R(U,V,X,Y) = \frac{r}{6}[g(U,X)g(V,Y) - g(U,V)g(X,Y)],$$

which shows that M is of constant curvature. Thus we can state as follows-

Theorem 2.1: *A 4-dimensional relativistic W_4 – flat space-time is an Einstein space-time and it is of constant curvature.*

It is known that a Lorentzian manifold of constant curvature is a manifold of conformally flat. Thus we have the following corollary

Corollary 2.1: *A 4-dimensional relativistic W_4 – flat space-time is a conformally flat space-time.*

In 1980, Kramer et al.[7] have been proved that a space is of O-type if the conformal curvature tensor vanishes on it. Thus we have a theorem as follows

Theorem 2.2: *A 4-dimensional relativistic W_4 – flat space-time is of O-type.*

Now, we consider a perfect fluid space-time with flat W_4-curvature tensor having Einstein's field equation in the presence of cosmological constant. Let $£_\xi$ be the Lie derivative operator along the vector field ξ generating the symmetry. The matter collineation defined by $(£_\xi T)(U,V)=0$ represents the symmetry of energy momentum tensor T. In view of equation (2.3), equation (1.3) takes the form

(2.5) $$(\lambda-\frac{r}{4})g(X,Y)=kT(X,Y).$$

If ξ be a Killing vector field on the space-time with W_4-flat curvature tensor, then

(2.6) $$(£_\xi g)(X,Y)=0.$$

Taking the Lie derivative of equation (2.5) along ξ, we obtain

(2.7) $$(\lambda-\frac{r}{4})(£_\xi g)(X,Y)=k(£_\xi T)(X,Y).$$

In virtue of equation (2.6), equation (2.7) shows that $(£_\xi T)(X,Y)=0,$ which shows that the space-time admits matter collineation. Conversely, If $(£_\xi T)(X,Y)=0,$ it follows that from equation (2.7), that ξ is Killing vector field. Hence we can state as follows:

Theorem 2.3: *If a 4-dimensional relativistic space-time having Einstein's field equation in the presence of cosmological constant W_4-flat curvature tensor, then the space-time satisfies matter collineation along a vector field ξ if and only if ξ is a Killing vector field.*

Next, Let us assume that the vector field ξ is a conformal Killing vector field, then we obtain

(2.8) $$(£_\xi g)(X,Y)=2\phi g(X,Y),$$

where ϕ is scalar, which is view of equation (2.7), gives

(2.9) $$(\lambda-\frac{r}{4})2\phi g(X,Y)=k(£_\xi T)(X,Y).$$

Using equation (2.5) in equation (2.9), we obtain

(2.10) $$(\pounds_{\xi} T)(X,Y) = 2\phi T(X,Y).$$

From above equation, we see that the energy-momentum tensor has Lie inheritance property along ξ. Conversely, if equation (2.10) holds, then it follows that equation (2.8) holds, i.e. the vector field ξ is a conformal Killing vector field. Thus we have a theorem as follows:

Theorem 2.4: *In a 4-dimensional relativistic space-time having Einstein's field equation in the presence of cosmological constant W_4 – flat curvature tensor, a vector field ξ is conformal Killing vector field if and only if the energy-momentum tensor T has a symmetry inheritance property along ξ.*

Now, Taking covariant derivative of equation (2.5), we obtain

(2.11) $$(\nabla_U T)(X,Y) = \frac{dr(U)}{4k} g(X,Y).$$

Since r is constant in a W_4 – curvature flat space-time, we have

(2.12) $$dr(U) = 0, \forall U.$$

Using equation (2.12) in equation (2.11), we obtain

(2.13) $$(\nabla_U T)(X,Y) = 0.$$

Thus we have a result as follows:

Theorem 2.5: *In a 4-dimensional relativistic W_4 – flat space-time having Einstein's field equation in presence of cosmological constant, the energy-momentum tensor T is covariant constant.*

Again, we consider a perfect fluid space-time with W_4 – flat curvature tensor having Einstein's field equation in the absence of cosmological constant. Using equations (2.3) and (1.4) in equation (1.1), we get

(2.14) $$-(r + k\rho) g(X,Y) = k(\rho + \sigma) A(X) A(Y),$$

which on contraction gives

(2.15) $$r=\frac{1}{4}k(\sigma-3\rho).$$

Now, taking $X=Y=\mu$ in equation (2.14}) and using $g(\mu,\mu)=-1$, we obtain

(2.16) $$r=k\sigma.$$

From equations (2.15) and (2.16), we have

(2.17) $$\sigma+\rho=0.$$

which gives that the perfect fluid behave as a cosmological constant.
Thus, in view of equation (2.17), equation (1.1) reduces to

(2.18) $$T(X,Y)=\rho g(X,Y).$$

For a W_4-flat space-time, the scalar curvature is constant. Thus σ is constant. Consequently, ρ is constant. Therefore, the covariant derivative of equation (2.18), we obtain

(2.19) $$(\nabla_U T)(X,Y)=0,$$

which shows that the energy momentum tensor is covariantly constant. Thus we have theorem as follows

Theorem 2.6: *In a 4-dimensional relativistic perfect fluid W_4-flat space-time following Einstein's field equation in the absence of cosmological constant, $\sigma+\rho=0$ and the isotropic pressure and energy density are constants. Moreover, energy momentum tensor is covariantly constant.*

Now, Taking the frame-field after contraction over X and Y of equation (1.4), we obtain

(2.20) $$r=-kt,$$

where t is $tr(T)$.

Therefore, equation (1.4) can be written as

$$S(X,Y) = k[T(X,Y) - \frac{t}{2} g(X,Y)]. \tag{2.21}$$

Einstein's field equation in the absence of cosmological constant for a purely electromagnetic distribution takes the form

$$S(X,Y) = kT(X,Y). \tag{2.22}$$

From equations (2.21) and (2.22), we obtained $t = 0.$ Thus from equation (2.20), we get $r = 0.$ Therefore, from equation (2.4), and we obtain $R(U,V,X,Y) = 0,$ which shows that the space is flat. Thus we arrive at

Theorem 2.7: *A 4-dimensional relativistic W_4 – flat space-time having Einstein's field equation in the absence of cosmological constant for a purely electromagnetic distribution is a Euclidean space.*

3. Space-time with Conservative W_4 – curvature Tensor

Definition 3.1: *A 4-dimensional relativistic space-time M is said to be W_4 – conservative if $(div\ W_4)(U,V)X = 0,$ where "div" denotes the divergence.*

From equation (1.6), we have

$$W_4(U,V)X = R(U,V)X + \frac{1}{n-1}[g(U,X)Q(V) - g(U,V)Q(X)], \tag{3.1}$$

where $Q(X)$ is Ricci operator.

The divergence of W_4 – curvature tensor is given b

$$(divW_4)(U,V)X = g((\nabla_{e_i} W_4)(U,V)X, e_i) = g((\nabla_{e_i} R)(U,V)X, e_i)$$

$$+ \frac{1}{n-1}[g(U,X)g((\nabla_{e_i} Q)V, e_i) - g(U,V)g(\nabla_{e_i} Q)(X, e_i)],$$

$$(divW_4)(U,V)X = (\nabla_U S)(V,X) - (\nabla_V S)(U,X) \tag{3.2}$$

$$+\frac{1}{6}[g(U,X)dr(V)-g(U,V)dr(X)].$$

If W_4 – curvature tensor is conservative then in virtue of equation (3.2), we obtain scalar curvature r is constant which implies that the Ricci tensor is of Codazzi type. Thus we can state as follows

Theorem 3.1: *A 4-dimensional relativistic space-time M admits a conservative W_4 – curvature tensor if and only if the Ricci tensor is Codazzi tensor. In both cases, the scalar curvature is constant.*

Now, Guifoyle and Nolan[8] named "Yang Pure Space" a 4-dimensional Lorentzian manifold (M,g) whose metric tensor solves Yang's equation $(\nabla_U S)(V,X)-(\nabla_V S)(U,X)=0.$ Thus we can state as follows

Theorem 3.2: *A 4-dimensional relativistic space-time M admits a conservative W_4 – curvature tensor is a Yang Pure space.*

Since we known that a 4-dimensional relativistic perfect fluid space-time with $\sigma+\rho\neq 0$ is a Yang Pure space-time if and only if space-time is RW space-time.

Theorem 3.3: A 4-dimensional relativistic space-time M admits a conservative W_4 – curvature tensor is a RW space-time.

It is known that divergence of conformal (Weyl) curvature tensor can be written as

(3.3)
$$(divC)(U,V)X=\frac{n-3}{n-2}[(\nabla_U S)(V,X)-(\nabla_V S)(U,X) \\ -\frac{1}{2(n-1)}\{g(V,X)dr(U)-g(U,X)dr(V)\}].$$

In virtue of equations (3.3), we observe that $(div\ C)(U,V)X=0$ if Ricci tensor is Codazzi tensor. Thus $(div\ W_4)(U,V)X=0.$
Thus we can state as follows

Theorem 3.4: *A 4-dimensional relativistic space-time M satisfying $(div\ C)(U,V)X=0$ if and only if $(div\ W_4)(U,V)X=0.$*

Since we known that a perfect fluid space- time of dimension ≥ 3 satisfies conservative conformal curvature tensor and the velocity vector field irrotational, then the space-time is a GRW space-time with $A\left(C\left(U,V\right)X\right)=0.$

Thus we can state as follows

Theorem 3.5: *A 4-dimensional relativistic space-time* M *satisfying* $\left(div\ C\right)\left(U,V\right)X=0$ *is a GRW space-time.*

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