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Projectively Flat Finsler Spaces with Special (α,β)-Metric

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Abstract: The (α,β)-metric is a Finsler metric which is constructed from a Riemannian metric α and a differential 1-form β. In this paper, we discuss the projective flatness of a Finsler space with a special $(\ ,\beta)$ metric in a locally Minkowski space.

Keywords: Finsler space, (α,β)-metric, projective flatness, locally Minkowski space, flat-parallel, associated Riemannian space.

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1. Introduction

A Finsler metric function $L(x,y)$ is called an (α,β) metric if L is a positively homogeneous function of a Riemannian metric $\alpha^2 = a_{ij}(x)\,y^i y^j$ and a differential 1-form $\beta = b_i y^i$ of degree one[1]. Interesting examples of (α,β)-metric are the Randers metric, Kropina metric and Matsumoto metric.

A Finsler space $F^n = (M^n, L)$ is called a locally Minkowski space[2] if M^n is covered by coordinate neighborhood system (x^i) in each of which L is a function of y^i only. A Finsler space $F^n = (M^n, L)$ is called projectively flat[3] if F^n is projective to a locally Minkowski space.

The condition for a Randers space to be projectively flat was given by Hashiguchi-Ichijyo[4] and Matsumoto[5]. The projective flatness of Kropina space was investigated by Matsumoto[5] and Matsumoto space was studied by Aikou- Hashiguchi-Yamauchi[6]. The condition for a Finsler space with a generalized Randers metric L satisfying $L^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^3$, where c_i's are constants, to be projectively flat was given by Park and Choi[7]. Recently, the projective flatness of Finsler spaces with some special metric has been studied in the paper[8]. A locally Minkowski space with (α,β)- metric is called flat-parallel if α is locally flat and β is parallel with respect to α[9].

The purpose of the present paper is to consider the projective flatness of Finsler spaces with a special (α,β)- metric $L = \alpha + \beta + \frac{\alpha}{\beta} + \frac{\alpha}{\beta}$.

2. Preliminaries

In a Finsler space $F^n = (M^n, L)$ with an $(\alpha,\beta$ -metric, let $\gamma^i_{jk}(x)$ be the Christoffel symbols constructed from the Riemannian metric a_{ij}. We denote by (;) the covariant differentiation with respect to $\gamma^i_{jk}(x)$. In a Finsler space F^n with (α,β)-metric, we define

$$\begin{aligned} &(2.1) \qquad 2r_{ij} = b_{i;j} + b_{j;i}, 2s_{ij} = b_{i;j} - b_{j;i}, s^i_j = a^{ir}s_{rj}, \\ &\qquad\qquad s_i = b^r s_{ri}, \gamma_{jhk} = a_{hr}\gamma^r_{jk}, b^2 = a^{rs}b_r b_s \end{aligned}$$

We shall denote the homogeneous polynomials in (y^i) of degree r by hp(r) for brevity. Now the following Matsumoto's theorem[5] is well-known.

Theorem 2.1. *A Finsler space F^n with an (α,β)-metric is projectively flat if and only if the space is covered by coordinate neighborhoods on which $\gamma^i_{jk}(x)$ satisfies*

$$\begin{aligned} (2.2) \qquad &\left(\gamma^i{}_{00} - \gamma_{000}y^i/\alpha^2\right)/2 + \left(\alpha L_\beta / L_\alpha\right)s^i{}_0 \\ &+ (L_{\alpha\alpha}/L_\alpha)(C + \alpha r_{00}/2\beta)\left(\alpha^2 b^i/\beta - y^i\right) = 0, \end{aligned}$$

where the subscript 0 means a contraction by y^i,

$$L_\alpha = \frac{\partial L}{\partial \alpha}, L_\beta = \frac{\partial L}{\partial \beta}, L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha}, L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta},$$

and C is given by

$$(2.3) \qquad C + \left(\alpha^2 L_\beta / \beta L_\alpha\right)s_0 + \left(\alpha L_{\alpha\alpha}/\beta^2 L_\alpha\right)\left(\alpha^2 b^2 - \beta^2\right)(C + \alpha r_{00}/2\beta) = 0.$$

By the homogeneity of L, we know $\alpha^2 L_{\alpha\alpha} = \beta^2 L_{\beta\beta}$.

Hence the formula (2.3) can be rewritten in the following form

$$\text{(2.4)} \quad \{1 + (L_{\beta\beta} / \alpha L_{\alpha\alpha})(\alpha^2 b^2 - \beta^2)\}(C + \alpha r_{00} / 2\beta) = (\alpha / 2\beta)\{r_{00} - (2\alpha L_\beta / L_\alpha) s_0\}$$

If $1 + (L_{\beta\beta} / \alpha L_{\alpha\alpha})(\alpha^2 b^2 - \beta^2) \neq 0$, then we can eliminate $(C + \alpha r_{00} / 2\beta)$ in (2.2) and it is written in the form

$$\text{(2.5)} \quad \{1 + L_{\beta\beta}(\alpha^2 b^2 - \beta^2)/(\alpha L_\alpha)\}\{(\gamma^i_{00} - \gamma_{000} y^i / \alpha^2)/2 + (\alpha L_\beta / L_\alpha) s^i_0\} + (L_{\alpha\alpha} / L_\alpha)(\alpha / 2\beta)\{r_{00} - (2\alpha L_\beta / L_\alpha) s_0\}(\alpha^2 b^i / \beta - y^i) = 0\,.$$

Thus, we have

Theorem 2.2. *If* $1 + (L_{\beta\beta} / \alpha L_{\alpha\alpha})(\alpha^2 b^2 - \beta^2) \neq 0$, *then a Finsler space* F^n *with an* (α, β)*-metric is projectively flat if and only if (2.5) is satisfied.*

It is known[10] that if α^2 contains as a factor, then the dimension is equal to two and $b^2 = 0$.

Throughout this paper, we assume that the dimension is more than two and $b^2 \neq 0$ that is, $\alpha^2 \not\equiv 0 \pmod{\beta}$.

3. Projectively Flat Space

Let F^n be a Finsler space with an (α, β)-metric given by

$$\text{(3.1)} \qquad L = \alpha + \beta + \frac{\alpha^2}{\beta} + \frac{\alpha^3}{\beta^2}\,.$$

It is known[1] that a Finsler space with (α, β)-metric given by (3.1) is flat-parallel if it is locally Minkowski.

In this section, we find the condition for a Finsler space F^n with the metric given by (3.1) to be projectively flat.

The partial derivatives with respect to α and β of metric (3.1) are given by

$$L_{\alpha}=1+\frac{2\alpha}{\beta}+\frac{3\alpha^{2}}{\beta^{2}},\ L_{\beta}=1-\frac{\alpha^{2}}{\beta^{2}}-\frac{2\alpha^{3}}{\beta^{3}},$$
$$L_{\alpha\alpha}=\frac{2}{\beta}+\frac{6\alpha}{\beta^{2}},\ L_{\beta\beta}=\frac{2\alpha^{2}}{\beta^{3}}+\frac{6\alpha^{3}}{\beta^{4}}. \tag{3.2}$$

If $1+\left(L_{\beta\beta}/\alpha L_{\alpha}\right)\left(\alpha^{2}b^{2}-\beta^{2}\right)=0,$ then we have

$$\beta^{4}-3\alpha^{2}\beta^{2}+2b^{2}\alpha^{3}\beta+6b^{2}\alpha^{4}=0$$

which leads to a contradiction.Thus Theorem 2.2 can be applied.

Substituting (3.2) into (2.1), we get

$$\left(\beta^{4}-3\alpha^{2}\beta^{2}+2b^{2}\alpha^{3}\beta+6b^{2}\alpha^{4}\right)\begin{Bmatrix}-4s_{0}^{i}\alpha^{6}-2\beta s_{0}^{i}\alpha^{5}+3\beta\gamma_{00}^{i}\alpha^{4}\\ +2\beta^{2}\left(\gamma_{00}^{i}+\beta s_{0}^{i}\right)\alpha^{3}\\ +\beta\left(\beta^{2}\gamma_{00}^{i}-3\gamma_{000}y^{i}\right)\alpha^{2}\\ -2\beta^{2}\gamma_{000}y^{i}\alpha-\beta^{3}\gamma_{000}y^{i}\end{Bmatrix} \tag{3.3}$$

$$+\begin{Bmatrix}24s_{0}\alpha^{8}+20s_{0}\alpha^{7}\beta+2\beta\left(2s_{0}\beta+9r_{00}\right)\alpha^{6}\\ +6\beta^{2}\left(3r_{00}-2s_{0}\beta\right)\alpha^{5}+2\beta^{3}\left(5r_{00}-2s_{0}\beta\right)\alpha^{4}\\ +2r_{00}\alpha^{3}\beta^{4}\end{Bmatrix}\left(\alpha^{2}b^{i}-\beta y^{i}\right)=0.$$

The equation (3.3) can be rewritten as a polynomial of tenth degree in α as follows

$$\left(p_{10}\alpha^{10}+p_{8}\alpha^{8}+p_{6}\alpha^{6}+p_{4}\alpha^{4}+p_{2}\alpha^{2}+p_{0}\right)$$
$$+\alpha\left(p_{9}\alpha^{8}+p_{7}\alpha^{6}+p_{5}\alpha^{4}+p_{3}\alpha^{2}+p_{1}\right)=0,$$

where

$$p_{10}=24\left(-s_0^ib^2+s_0b^i\right),\ p_9=20\left(-s_0^ib^2+s_0b^i\right)\beta,$$

$$p_8=\begin{pmatrix}12s_0^i\beta^2-4b^2s_0^i\beta^2+18b^2\beta\gamma_{00}^i\\ -24s_0\beta y^i+4s_0\beta^2b^i+18r_{00}\beta b^i\end{pmatrix},$$

$$p_7=\begin{pmatrix}6s_0^i\beta^3+18b^2\beta^2\gamma_{00}^i+12b^2\beta^3s_0^i\\ -20s_0\beta^2y^i+18r_{00}\beta^2b^i-12s_0\beta^3b^i\end{pmatrix},$$

$$p_6=\begin{pmatrix}-4s_0^i\beta^4-9\beta^3\gamma_{00}^i+10b^2\beta^3\gamma_{00}^i+4b^2\beta^4s_0^i-18b^2\beta\gamma_{000}y^i\\ -4s_0\beta^3y^i-18r_{00}\beta^2b^i+10r_{00}\beta^3b^i-4s_0\beta^4b^i\end{pmatrix},$$

$$p_5=\begin{pmatrix}-8s_0^i\beta^5-6\beta^4\gamma_{00}^i+2b^2\beta^4\gamma_{00}^i-18b^2\beta^2\gamma_{000}y^i\\ -18r_{00}\beta^3y^i+12s_0\beta^4y^i-2r_{00}\beta^4b^i\end{pmatrix},$$

$$p_4=\begin{pmatrix}9\beta^3\gamma_{000}y^i-10b^2\beta^3\gamma_{000}y^i\\ -10r_{00}\beta^4y^i+4s_0\beta^5y^i\end{pmatrix},$$

$$p_3=\begin{pmatrix}2\beta^4\gamma_{00}^i+2\beta^7s_0^i+6\beta^4\gamma_{000}y^i\\ -2b^2\beta^4\gamma_{000}y^i-2r_{00}\beta^5b^i\end{pmatrix},$$

$$p_2=\beta^7\gamma_{00}^i,\ p_1=-2\beta^6\gamma_{000}y^i,\ p_0=-\beta^7\gamma_{000}y^i.$$

Since $p_{10}\alpha^{10}+p_8\alpha^8+p_6\alpha^6+p_4\alpha^4+p_2\alpha^2+p_0$ and $p_9\alpha^8+p_7\alpha^6+p_5\alpha^4+p_3\alpha^2+p_1$ are rational and α is irrational in y^i, we have

(3.4) $$p_{10}\alpha^{10}+p_8\alpha^8+p_6\alpha^6+p_4\alpha^4+p_2\alpha^2+p_0=0,$$

(3.5) $$p_9\alpha^8+p_7\alpha^6+p_5\alpha^4+p_3\alpha^2+p_1=0.$$

It follows from (3.4) that the term which has a factor β is $24\left(-s_0^ib^2+s_0b^i\right)\alpha^{10}$. Since $\alpha^2\not\equiv 0\ (\mathrm{mod.}\beta)$, we have a vector $\lambda^i=\lambda^i(x)$ satisfying $s_0b^i-b^2s_0^i=\lambda^i\beta.$

Transvecting this by $y_i=a_{ij}y^j$, we get $s_0=\lambda^iy_i$ so that, $\lambda_i=s_i$.

Therefore, we have, $b^2s_0^i=s_0b^i-s^i\beta$, that is,

(3.6) $$b^2s_{ij}=b_is_j-b_js_i.$$

Secondly, we observe in (3.5) that the term which must have a factor α^2 is $-2\beta^6\gamma_{000}y^i$. Hence we have 1-form $v_0 = v_i(x)y^i$ such that

(3.7) $$\gamma_{000} = v_0\alpha^2 .$$

From (3.4) and (3.7), the term which has a factor α^2 is $\left(\gamma^i_{00} - v_0 y^i\right)\beta^7$.
Hence we have $\mu^i = \mu^i(x)$ satisfying

(3.8) $$\gamma^i_{00} - v_0 y^i = \mu^i\alpha^2 .$$

Transvecting (3.8) by y_i, we have from (3.7), $\mu^i y_i = 0$, which implies $\mu^i = 0$.

Thus, we have

(3.9) $$\gamma^i_{00} = v_0 y^i ,$$

that is,

(3.10) $$2\gamma^i_{jk} = v_k\delta^i_j + v_j\delta^i_k ,$$

which shows that the Riemannian space is projectively flat.

Next substituting (3.7) and (3.9) into (3.3), we have

(3.11) $$\left(\beta^4 - 3\alpha^2\beta^2 + 2b^2\alpha^3\beta + 6b^2\alpha^4\right)\left(-2\alpha^3 - \beta\alpha^2 + \beta^3\right)s^i_0$$
$$+\left\{\begin{array}{l} 12s_0\alpha^5 + 10s_0\beta\alpha^4 + \beta\left(2s_0\beta + 9r_{00}\right)\alpha^3 \\ +3\beta^2\left(3r_{00} - 2s_0\beta\right)\alpha^2 + \beta^3\left(5r_{00} - 2s_0\beta\right)\alpha \\ +r_{00}\beta^4 \end{array}\right\}\left(\alpha^2 b^i - \beta y^i\right) = 0.$$

Transvecting (3.11) by b_i, we get

(3.12) $$\left\{\begin{array}{l} \left(-6s_0\beta + 9r_{00}b^2\right)\alpha^4 + \left(5r_{00}b^2\beta^2 - 4s_0\beta^3 - 9r_{00}\beta^2\right)\alpha^2 \\ +\left(2s_0\beta^5 - 5r_{00}\beta^4\right) \end{array}\right\}\alpha$$
$$+\left\{\begin{array}{l} \left(-7s_0\beta + 9r_{00}b^2\right)\beta\alpha^4 + \left(2s_0\beta - 9r_{00} + r_{00}b^2\right)\alpha^2\beta^3 \\ +\left(s_0\beta - r_{00}\right)\beta^5 \end{array}\right\} = 0,$$

which implies

$$\begin{matrix}\left(-6s_0\beta+9r_{00}b^2\right)\alpha^4+\left(5r_{00}b^2\beta^2-4s_0\beta^3-9r_{00}\beta^2\right)\alpha^2 \\ +\left(2s_0\beta^5-5r_{00}\beta^4\right)\end{matrix}=0.$$

The term which has a factor α^2 is $\left(2s_0\beta^5-5r_{00}\beta^4\right)$ i.e., $2s_0\beta^5-5r_{00}\beta^4=0$
This implies $2s_0\beta=5r_{00}$.

Thus, the above equation becomes

(3.13) $$\left(-6s_0\beta+9r_{00}b^2\right)\alpha^4+\left(5r_{00}b^2\beta^2-4s_0\beta^3-9r_{00}\beta^2\right)\alpha^2=0.$$

Therefore there exists a function $k=k(x)$ such that

(3.14) $$-6s_0\beta+9r_{00}b^2=k\beta^2,\ 5r_{00}b^2-4s_0\beta-9r_{00}=k\alpha^2.$$

Eliminating r_{00} from (3.14), we have

(3.15) $$\left(6b^2+54\right)s_0\beta=k\left(5b^2\beta^2-9b^2\alpha^2-9\beta^2\right),$$

i.e.

(3.16) $$\left(6b^2+54\right)\left(s_ib_j+s_jb_i\right)=2k\left\{\left(5b^2-9\right)b_ib_j-9b^2a_{ij}\right\}.$$

Transvecting (3.16) by a^{ij}, we have

(3.17) $$kb^2\left(5b^2-9n-9\right)=0.$$

which implies k= 0. Thus from (3.15), we have $s_0=0$, and hence from (3.14) we obtain $r_{00}=0$, i.e., $r_{ij}=0$.

On the other hand, from $s_i=0$ and (3.6) we have $s_{ij}=0$.

So we get $b_{i;j}=0$.

Conversely, it is easy to see that (3.6) is a consequence of (3.9) and $b_{i;j}=0$. Thus we have

Theorem 3.1. *A Finsler space* $F^n\,(n>2)$ *with the* (α,β)*-metric given by (3.1) is projectively flat if and only if the associated Riemannian space* (M^n,α) *is projectively flat and* $b_{i;j}=0.$

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