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Modelling and Analysis of the Role of Vaccination and Isolation on the Spread of Infectious Diseases

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Abstract: In this paper, a four dimensional SIS epidemic non-linear model with immigration is proposed and analyzed to study the role of vaccination and isolation on the spread of an infectious diseases. The model has been analyzed by using stability theory of differential equations and simulation. The model has two equilibria namely, disease free and non-trivial endemic equilibrium.

It is shown that the disease free equilibrium is always unstable and the endemic equilibrium, if exists, becomes locally as well as non-linearly stable under certain conditions. This analysis also implies that as the rate of vaccination of susceptible human population density or the rate of isolation of severely infected human population density increases, the spread of infectious disease decreases. A numerical analysis of the model is also performed which supports the analytical results.

Keywords: Vaccination, Isolation, Stability, Lyapunov's function.

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1. Introduction

Infectious diseases kill more people worldwide than any other single reason. These diseases are considered contagious or communicable, meaning that they can be passed from person to person by direct contact. It is also possible for such diseases to spread indirectly through unhygienic conditions, or from animals to people, in which case they are called zoonotic diseases.

A variety of agents (pathogens) can be responsible for the spread of infectious diseases, including viruses, bacteria, fungi, protozoans, etc. With these infectious organisms, there are various modes of transmission, direct as well as indirect, for the spread of infectious diseases. In order to treat an infectious disease, doctors must be able to find out the source of infection and treat it appropriately. Many infectious diseases make the body vulnerable to secondary infections, in which case other organisms move in to take advantage of a weakened immune system and cause diseases [5,9].

Vaccination is the administration of a vaccine to stimulate the immune system of an individual to develop protective immunity to a disease. It is considered to be the most effective method of controlling infectious diseases. The use of vaccination has been widely studied and verified; for example, the influenza vaccine, TB vaccine and the chicken pox vaccine among others. Smallpox was likely the first disease people tried to prevent by purposely inoculating themselves with other types of infections. No public health tool has been as successful and effective as vaccines at saving lives, particularly among the world's children [1,2,6,7].

One other method of control of an infectious disease such as smallpox began even before vaccination, is isolation. From early times, first with leprosy and then with plague, it has been noted that it might be possible to avoid certain diseases by making sure that no direct or indirect contact occurs between infected and susceptible persons. The concept of starting sanatorium for diseases such as TB is used now by keeping this idea in view. Further, even in hospitals now, separate wards exist for infectious diseases [3,8].

In view of the above, in this paper, therefore, effects of vaccination and isolation on the spread of infectious diseases are modeled and analyzed by using stability theory of differential equations and numerical simulation. We illustrate these effects in the case of an SIS model.

2. An SIS Model

Let $X(t)$, $Y(t)$, $V(t)$ and $Z(t)$ denote densities of susceptible, infective, vaccinated and isolated classes respectively of total human population density $N(t)$, in a region under consideration. By assuming simple mass action interaction, an SIS model of the situation can be written as:

$$
\begin{aligned}
X' &= A - \beta XY - eX - \phi X + \nu Y + kV \\
Y' &= \beta XY - (\nu + \alpha + e + \psi)Y \\
V' &= \phi X - eV - kV \\
Z' &= \psi Y - (e + \alpha)Z
\end{aligned}
\tag{1}
$$

with initial conditions $X(0) > 0, Y(0) \geq 0, V(0) \geq 0, Z(0) \geq 0.$

Here $()'$ denotes $d()/dt$.

In the modeling process, the following assumptions are made

(i) The rate of vaccinated people is proportional to *X*.
(ii) The rate of isolated people is proportional to *Y*.
(iii) The birth and natural death rates of host population are equal.

List of coefficients in the model (1):

A : Immigration rate of susceptible human population from outside the region
e : Rate of emigration of host population
β : Transmission coefficient due to infective human population
ϕ : Rate of vaccination of susceptible human population
ν : Recovery rate of infected human population
α : Disease related death rate
ψ : Rate of isolation of severely infected human population
k : Rate by which vaccinated human individuals become susceptible

All the coefficients in the model (1) are assumed to be positive constants.
In the following we analyze the model (1) by using stability theory of differential equations.

3. Equilibrium Analysis

Since $X + Y + V + Z = N$, the model (1) can be rewritten as follows:

$$
\begin{aligned}
Y' &= \beta (N - Y - V - Z)Y - (\nu + \alpha + e + \psi)Y \\
V' &= \phi (N - Y - V - Z) - (e + k)V \\
Z' &= \psi Y - (e + \alpha)Z \\
N' &= A - eN - \alpha Y - \alpha Z
\end{aligned}
\tag{2}
$$

The following lemma establishes region of attraction for the system (2).

Lemma : *The set*

$$\Omega = \left\{ \begin{array}{l} \left(Y, V, Z, N : 0 \le Y \le N \le \dfrac{A}{e}, 0 < V \le \dfrac{\varphi A}{e(\varphi + e + k)}, \right. \\ 0 \le Z \le \dfrac{\psi A}{e e + \alpha)}, \dfrac{A}{e + \alpha} \le N \le \dfrac{A}{e} \end{array} \right\} \textit{attracts}$$

all the solutions initiating in the positive orthant.

Proof: Here we give only a brief outline of the proof, the detail proof can be seen in [4].

From the last equation of model (2), we have

$$N' = A - eN - \alpha Y - \alpha Z \le A - eN$$

and $$N' = A - eN - \alpha Y - \alpha Z \ge A - (e + \alpha)N$$

By using comparison theorem, we get

$$\frac{A}{e + \alpha} \le N \le \frac{A}{e}.$$

From the second equation of model (2), we have

$$V' \le \phi \frac{A}{e} - (\phi + e + k)V$$

which gives

$$0 < V \le \frac{\phi A}{e(\phi + e + k)}$$

Similarly, from the equation for isolated population in (2), we have

$$0 \le Z \le \frac{\psi A}{e(e + \alpha)}.$$

The result of equilibrium analysis of the model (2) are stated in the following theorem:

Theorem 1: *The system (2) has following two equilibria:*

(i) $\overline{E}\left(0, \overline{V}, 0, \overline{N}\right)$, *the disease free equilibrium.*

where $\overline{V} = \dfrac{\phi A}{e(\phi+e+k)}$ *and* $\overline{N} = \dfrac{A}{e}$;

(ii) $E^*(Y^*, V^*, Z^*, N^*)$*, the endemic equilibrium.*

Proof: The existence of $\overline{E}$ is obvious. Here we prove the existence of E^*.The equilibrium point E^* is given as the solutions of system of following equations, which are obtained after some simplification from (2) by equating left hand sides to zero:

$$\beta(N-Y-V-Z)-(\nu+\alpha+e+\psi)=0 \tag{3}$$

$$\phi(N-Y-Z)-(\phi+e+k)V=0 \tag{4}$$

$$\psi Y-(e+\alpha)Z=0 \tag{5}$$

$$A-eN-\alpha Y-\alpha Z=0 \tag{6}$$

Now eliminating Z between equations (5) and (6) we get

$$\frac{(\psi+e+\alpha)}{(e+\alpha)}Y=\left(\frac{A-eN}{\alpha}\right) \tag{7}$$

Now using (4) and (5) in (3), we get

$$N-\frac{(\psi+e+\alpha)}{(e+\alpha)}Y-\frac{(\nu+\alpha+e+\psi)(\phi+e+k)}{\beta(e+k)}=0 \tag{8}$$

Using (7) in (8), we get

$$F(N)=N-\left(\frac{A-eN}{\alpha}\right)-\frac{(\nu+\alpha+e+\psi)(\phi+e+k)}{\beta(e+k)}=0 \tag{9}$$

Now

$$F\left(\frac{A}{\alpha+e}\right)=-\frac{(\nu+\alpha+e+\psi)(\phi+e+k)}{\beta(e+k)}<0 \tag{10}$$

$$F\left(\frac{A}{e}\right)=\frac{A}{e}-\frac{(\nu+\alpha+e+\psi)(\phi+e+k)}{\beta(e+k)} \tag{11}$$

Here $F\left(\frac{A}{e}\right) > 0$ provided $R_0 = \frac{\beta A(e+k)}{e\,(\nu+\alpha+e+\psi)(\phi+e+k)} > 1$; where R_0 is the basic reproduction number.

Thus, it is clear that there exists a root N^* of $F(N) = 0$ in the interval $\frac{A}{e+\alpha} \le N \le \frac{A}{e}$, provided $R_0 > 1$. Further, this root will be unique as $F'(N) = 1 + \frac{e}{\alpha} > 0$.

Remark: We can check that at E^* , $\frac{dY}{d\phi} < 0$ and $\frac{dY}{d\psi} < 0$. These conditions imply that, as the rate of vaccination of susceptible human population and rate of isolation of severely infected human population increases, the density of infectives decreases. These can be checked as follows

From (3) and (4), we get

$$V = \frac{\phi(\nu+\alpha+e+\psi)}{\beta(e+k)} \tag{12}$$

From (5), we get

$$Z = \frac{\psi}{(e+\alpha)} Y \tag{13}$$

which after using (6) gives us

$$N = \frac{1}{e}\left[A - \frac{\alpha(\psi+e+\alpha)}{(e+\alpha)} Y\right] \tag{14}$$

Using these values in (4), we get

$$Y\left[\frac{\alpha(\psi+e+\alpha)}{e(e+\alpha)} + \frac{\psi}{(e+\alpha)} + 1\right] = \frac{A}{e} - \frac{(\phi+e+k)(\nu+\alpha+e+\psi)}{\beta(e+k)} \tag{15}$$

On differentiating (15) with respect to ϕ, we get

$$\frac{dY}{d\phi}\left[\frac{\alpha(\psi+e+\alpha)}{e(e+\alpha)}+\frac{\psi}{(e+\alpha)}+1\right]=-\frac{(\nu+\alpha+e+\psi)}{\beta(e+k)} \tag{16}$$

This clearly shows that at E^*,

$$\frac{dY}{d\phi}<0$$

Thus it is seen here that as the rate of vaccination of susceptible human population density increases, the infective human population density decreases at E^*.

In the similar manner, we can show that at E^*, $\frac{dY}{d\psi}<0$. Thus it is seen here that as the rate of isolation of severely infected human population density increases, the infective human population density decreases at the equilibrium point E^*.

4. Stability Analysis

Now we shall study the stability behavior of above equilibria. The local stability result of equilibrium points $\overline{E}$ and E^* are given in the following theorem:

Theorem 2: *The equilibria $\overline{E}$ is locally unstable if $R_0>1$ and the equilibrium E^* is locally asymptotically stable provided the following conditions are satisfied*

$$\left(\beta Y^*+k_1\phi\right)^2<\frac{4}{3}k_1\left(\beta Y^*\right)(\phi+e+k) \tag{17}$$

$$\alpha\psi<d(e+\alpha) \tag{18}$$

where $k_1=\frac{1}{3}\frac{(\phi+e+k)\beta Y^*}{\phi^2}\min\left\{\frac{(e+\alpha)}{\psi},\frac{e}{\alpha}\right\}$

Proof: The local stability behavior of $\overline{E}$ can be studied by computing corresponding variational matrix for system (2) and for nontrivial equilibrium point E^*, it can be studied by using Lyapunov's theory.

The variational matrix M_i corresponding to equilibrium points is given by:

$$M_i = \begin{bmatrix} \beta(N-V-Z)-2\beta Y-(\nu+\alpha+e+\psi) & -\beta Y & -\beta Y & \beta Y \\ -\phi & -(\phi+e+k) & -\phi & \phi \\ \psi & 0 & -(e+\alpha) & 0 \\ -\alpha & 0 & -\alpha & -e \end{bmatrix}$$

Local Stability Behavior of $\overline{E}\left(0,\overline{V},0,\overline{N}\right)$:

The variational matrix corresponding to equilibrium point $\overline{E}$ is given by

$$\overline{M} = \begin{bmatrix} \beta(\overline{N}-\overline{V})-(\nu+\alpha+e+\psi) & 0 & 0 & 0 \\ -\phi & -(\phi+e+k) & -\phi & \phi \\ \psi & 0 & -(e+\alpha) & 0 \\ -\alpha & 0 & -\alpha & -e \end{bmatrix}$$

Here $\beta(\overline{N}-\overline{V})-(\nu+\alpha+e+\psi)$ is one of the eigen values of the above variational matrix, which will be positive provided $R_0 = \dfrac{\beta A(e+k)}{e\,(\nu+\alpha+e+\psi)(\phi+e+k)} > 1$. Hence $\overline{E}$ is unstable if $R_0 > 1$.

Local Stability behaviour of $E^*\left(Y^*,V^*,Z^*,N^*\right)$:

We study the stability behaviour of E^* by Lyapunov's method. For this we linearise the system (2) by using transformations

$$y = Y - Y^*,\ v = V - V^*, z = Z - Z^*, n = N - N^*$$

and the following positive definite function to find the sufficient conditions for stability

$$V = \frac{1}{2}y^2 + \frac{k_1}{2}v^2 + \frac{k_2}{2}z^2 + \frac{k_3}{2}n^2 \tag{19}$$

(where k_1, k_2 and k_3 are positive constants to be chosen appropriately).

Differentiating (19) with respect to t and using the linearized version of (2), $\dfrac{dV}{dt}$ can be written as:

$$V' = -(\beta Y^*)y^2 - k_1(\phi+e+k)v^2 - k_2(e+\alpha)z^2 - (k_3 e)n^2$$

$$+\left(-\beta Y^{*}-k_{1}\phi\right)yv+\left(-\beta Y^{*}+k_{2}\psi\right)yz++\left(\beta Y^{*}-k_{3}\alpha\right)yn+\left(-k_{1}\phi\right)vz$$

$$+\left(k_{1}\phi\right)vn+\left(-k_{3}\alpha\right)zn$$

Choosing $k_2 = \frac{\beta Y^*}{\psi}$ and $k_3 = \frac{\beta Y^*}{\alpha}$, we get

$$V' = -\left[\left(\beta Y^*\right)y^2 + \left(\beta Y^* + k_1\phi\right)yv + \frac{k_1}{3}(\phi + e + k)v^2\right]$$

$$-\left[\frac{k_1}{3}(\phi + e + k)v^2 + \left(k_1\phi\right)vz + \frac{k_2}{2}(e + \alpha)z^2\right]$$

$$-\left[\frac{k_1}{3}(\phi + e + k)v^2 - \left(k_1\phi\right)vn + \frac{k_3 e}{2}n^2\right] -\left[\frac{k_2}{2}(e + \alpha)z^2 + \left(k_3\alpha\right)zn + \frac{k_3 e}{2}n^2\right]$$

Now the conditions for $\frac{dV}{dt}$ to be negative definite can be written as:

(20) $$\left(\beta Y^* + k_1\phi\right)^2 < \frac{4}{3}k_1\left(\beta Y^*\right)(\phi + e + k)$$

(21) $$k_1 < \frac{2}{3}\frac{(\phi + e + k)(e + \alpha)}{\phi^2}\frac{\beta Y^*}{\psi}$$

(22) $$k_1 < \frac{2}{3}\frac{(\phi + e + k)e}{\phi^2}\frac{\beta Y^*}{\alpha}$$

(23) $$\alpha\psi < e\,(e + \alpha)$$

Now if we choose $k_1 = \frac{1}{3}\frac{(\phi + e + k)\beta Y^*}{\phi^2}\min\left\{\frac{(e + \alpha)}{\psi}, \frac{e}{\alpha}\right\}$, then inequality (21) and (22) will satisfy automatically. Hence V' is negative definite if (20) and (23) are satisfied. Thus, E^* is locally stable if (17) and (18) are satisfied. The nonlinear stability results for E^* are given by the following theorem:

Theorem 3: *The equilibrium point E^* is nonlinearly asymptotically stable in Ω provided the following inequalities are satisfied:*

(24) $$(\beta + m_1\phi)^2 < \frac{4}{3} m_1 \beta (\phi + e + k)$$

(25) $$\alpha\psi < e(e+\alpha)$$

where $m_1 = \frac{1}{3}\frac{(\phi+e+k)\beta}{\phi^2}\min\left\{\frac{(e+\alpha)}{\psi}, \frac{e}{\alpha}\right\}$. We can prove the above theorem by using the following positive definite function:

(26)
$$V = \left(Y - Y^* - Y^* \ln\frac{Y}{Y^*}\right) + \frac{m_1}{2}\left(V - V^*\right)^2 + \frac{m_2}{2}\left(Z - Z^*\right)^2 + \frac{m_3}{2}\left(N - N^*\right)^2,$$

where m_1, m_2 and m_3 are positive constants to be chosen appropriately.

5. Numerical Simulation

Here we discuss the existence and stability of the nontrivial equilibrium point E^* by taking the following set of parameter values:

$A = 1000$, $e = 0.02$, $\alpha = 0.03$, $\beta = 0.000005$, $\nu = 0.05$, $\lambda = 0.000001$, $\nu_1 = 0.02$, $\phi = 0.09$, $\psi = 0.01$, $k = 0.2$

For these values of parameters, the value of nontrivial equilibrium point E^* corresponding to (2) is obtained as follows (using MAPLE):

$$Y^* = 6909,\ V^* = 9000,\ Z^* = 690,\ N^* = 38600$$

The variational matrix corresponding to the equilibrium point E^* is obtained as:

$$M^* = \begin{bmatrix} -0.03454 & -0.03454 & -0.03454 & 0.03454 \\ -0.09 & -0.31 & -0.09 & 0.09 \\ 0.01 & 0 & -0.1 & 0 \\ -0.03 & 0 & -0.03 & -0.02 \end{bmatrix}$$

The eigen values of this matrix are:

$$-0.0985,\ -0.0233+0.0279\,i,\ \ -0.0233-0.0279\,i,\ \ -0.3192.$$

All of which are with negative real part. Hence E^* is locally stable.

Now numerical simulation is performed for Y vs. N for the different initial starts and displayed in the figure 1 which indicates nonlinear stability of the point $\left(N^*,Y^*\right)$ in $N-Y$ plane. The model (2) has also been solved by MAPLE and Graphs of the variable Y with respect to t for various values of parameters ϕ and ψ have been plotted in figure 2 and figure 3. From these, following observations have been drawn:

(i) From figure 2, it is noted that $Y(t)$ decreases as ϕ (rate of vaccination of susceptible human population) increases.

(ii) From figure 3, it is noted that $Y(t)$ decreases as ψ (rate of isolation of severely infected human population) increases.

These results are expected as increase (decrease) in the vaccination and isolation causes decrease (increase) of the density of infectives.

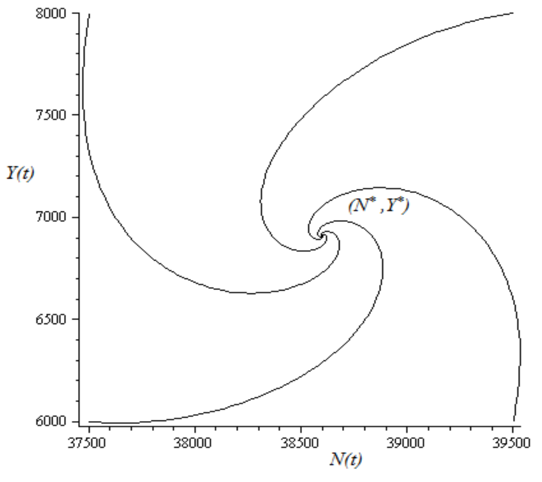

Fig. 1 Phase plots between susceptible human population density N and infected human population density Y

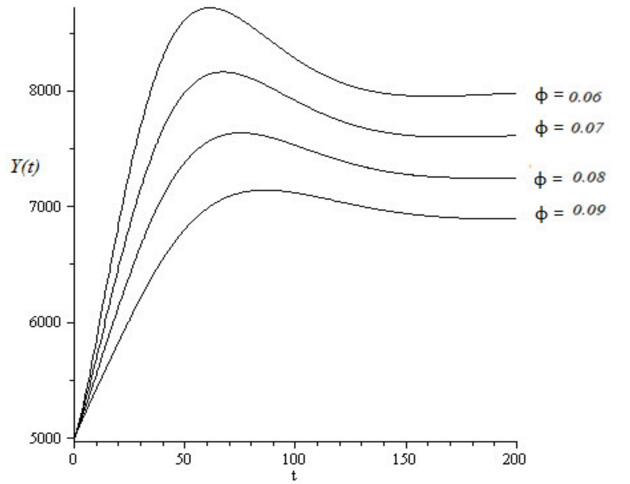

Fig. 2 Plots of $Y(t)$ w.r.t. t for different values of rate of vaccination of susceptible human population ϕ

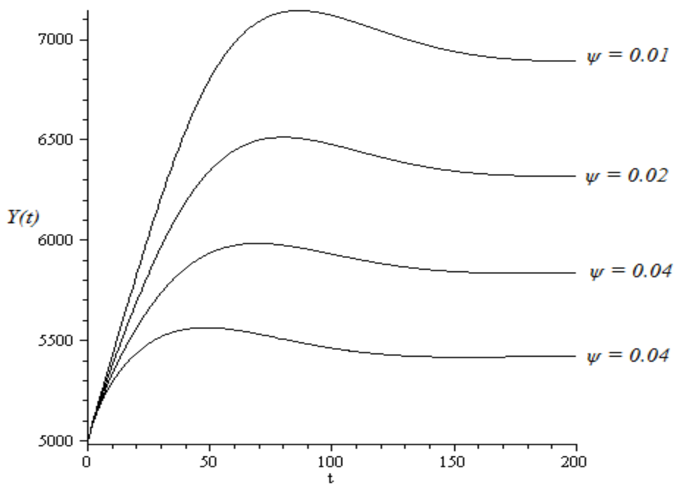

Fig. 3 Plots of $Y(t)$ for different values of rate of isolation of severely infected human population ψ

6. Conclusions

In this paper, a four dimensional SIS non-linear model with immigration has been proposed and analyzed to study the effect of vaccination and isolation on the spread of infectious diseases. The following assumptions have been made in the modeling process,

(a) The rate of vaccination is proportional to the density of susceptibles.

(b) The rate of isolated people is proportional to the density of infectives.

The model has been analyzed analytically as well as by computer simulation. The following main conclusions have been drawn from the analysis:

(a) The density of infectives decreases as rate of vaccination of susceptible human population increases.

(b) The density of infectives decreases, as the rate of isolation of severely infected people from active human population increases.

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