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Study of Generalized Lorentzian Para-Sasakian Manifolds

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Abstract: In this paper, we have studied the generalized Lorentzian Para-Sasakian manifold with concircularly flat and quasi concircularly flat curvature tensor.We have shown that a generalized Lorentzian Para-Sasakian manifold cannot be Semi-Pseudo symmetric manifold $(SPS)_n$.

Keywords and Phrases: Lorentzian Para-Sasakian manifold, locally isometric, Semi-Pseudo symmetric manifold.
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1. Preliminaries

In 1988 K. Matsumoto[1] introduced the notion of Lorentzian Para-Sasakian manifold. I. Mihai and R. Rosca[2] defined the same notion and thereafter some authors[3] studied Lorentzian Para-Sasakian manifold. We have studied generalized Lorentzian Para-Sasakian manifold in which

(a) $C = 0$, where C is concircular curvature tensor.

(b) $\tilde{C} = 0$, where $\tilde{C}$ is quasi concircular curvature tensor.

In both the cases, it is shown that generalized Lorentzian Para-Sasakian manifold is isometric with a unit sphere $S^n(1)$.

(c) $R(X,Y).C = 0$ has been discussed where $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y.

2. Generalized Lorentzian Para-Sasakian Manifolds

A differentiable manifold of dimension n is called generalized Lorentzian Para-Sasakian manifold, if it admits a (1,1) tensor field ϕ ,two contravariant vector fields ξ_1 and ξ_2 ,two 1-forms η_1 and η_2 and a Lorentzian metric g which satisfy

(2.1) $$\eta_i(\xi_j) + \delta_{ij} = 0,$$

(2.2) $$\phi^2 X = I + \eta_1(X)\xi_1 + \eta_2(X)\xi_2,$$

(2.3) $$g(\phi X, \phi Y) = g(X,Y) + \eta_1(X)\eta_1(Y) + \eta_2(X)\eta_2(Y),$$

(2.4) (i) $$g(X,\xi_1) = \eta_1(X), \qquad g(X,\xi_2) = \eta_2(X),$$

(2.4) (ii) $$\nabla_X \xi_1 = \phi X - \xi_2, \qquad \nabla_X \xi_2 = \phi X - \xi_1,$$

(2.5) $$2(\nabla_X \phi)Y = [g(X,Y) + \eta_1(X)\eta_1(Y) + \eta_2(X)\eta_2(Y)](\xi_1 + \xi_2)$$
$$+[X + \eta_1(X)\xi_1 + \eta_2(X)\xi_2][\eta_1(Y) + \eta_{21}(Y)].$$

On such generalized Lorentzian Para-Sasakian manifold the following relations hold

(2.6) $$\phi\xi_1 = 0, \quad \phi\xi_2 = 0, \quad \eta_1(\phi X) = 0, \quad \eta_2(\phi X) = 0,$$

(2.7) Rank $\phi = n - 2.$

Also a generalized Lorentzian Para-Sasakian manifold M is said to an η-Einstein if its Ricci tensor S is of the form

(2.8) $$S(X,Y) = ag(X,Y) + b\eta_1(X)\eta_1(Y) + c\eta_2(X)\eta_2(Y),$$

for any vector fields X, Y ,where a, b and c are functions on M. Further, on such a generalized Lorentzian Para-Sasakian manifold with $(\phi, \xi_1, \xi_2, \eta_1, \eta_2, g)$ structure, the following relations hold

(i) $$g(R(X,Y)Z, \xi_1) = \eta_1(R(X,Y)Z) = g(Y,Z)\eta_1(X) - g(X,Z)\eta_1(Y),$$

(2.9) (ii) $g(R(X,Y)Z,\xi_2) = \eta_2(R(X,Y)Z) = g(Y,Z)\eta_2(X) - g(X,Z)\eta_2(Y)$,

(2.10) (i) $R(\xi_1, X)Y = g(X,Y)\xi_1 - \eta_1(Y)X$,

(ii) $R(\xi_2, X)Y = g(X,Y)\xi_2 - \eta_2(Y)X$,

(2.11) (i) $R(\xi_1, X)\xi_1 = \eta_1(X)\xi_1 + X$,

(ii) $R(\xi_2, X)\xi_2 = \eta_2(X)\xi_2 + X$,

(2.12) (i) $R(X,Y)\xi_1 = -\eta_1(Y)X + \eta_1(X)Y$,

(ii) $R(X,Y)\xi_2 = -\eta_2(Y)X + \eta_2(X)Y$,

(2.13) (i) $S(X,\xi_1) = (n-1)\eta_1(X)$,

(ii) $S(X,\xi_2) = (n-1)\eta_2(X)$,

(2.14) $S(\phi Y, \phi Z) = S(Y,Z) + (n-1)\eta_1(Y)\eta_1(Z) + (n-1)\eta_2(Y)\eta_2(Z)$,

(2.15) (i) $(\nabla_X \eta_1)(Y) = g(X, \phi Y)$,

(ii) $(\nabla_X \eta_2)(Y) = g(X, \phi Y)$.

For any vector fields X, Y, Z and $R(X,Y)Z$ is the Riemannian curvature tensor. Putting (2.3) in (2.8) we get

(2.16) $$S(X,Y) = ag(X,Y) + b[g(\phi X, \phi Y) - g(X,Y) - \eta_2(X)\eta_2(Y)] \\ + c\eta_2(X)\eta_2(Y),$$

(2.17) $S(X,Y) = (a-b)g(X,Y) + bg(\phi X, \phi Y) + (c-b)\eta_2(X)\eta_2(Y)]$.

If we take $a-b=d$ and $-b=e$, then (2.17) becomes

(2.18) $S(X,Y) = dg(X,Y) + bg(\phi X, \phi Y) + e\eta_2(X)\eta_2(Y)]$.

By virtue of (2.8) and (2.14) we get

$$S(\phi X,\phi Y)=(n-1)\eta_1(X)\eta_1(Y)+(n-1)\eta_2(X)\eta_2(Y)+ag(X,Y)$$
$$+b\eta_1(X)\eta_1(Y)+c\eta_2(X)\eta_2(Y),$$

$$S(\phi X,\phi Y)=ag(X,Y)+[b+n-1]\eta_1(X)\eta_1(Y)+[c+n-1]\eta_2(X)\eta_2(Y),$$

(2.19) $$S(\phi X,\phi Y)=ag(X,Y)+f[\eta_1(X)\eta_1(Y)]+h[\eta_2(X)\eta_2(Y)],$$

where $e=b+n-1, h=c+n-1$.

3. Generalized Lorentzian Para-Sasakian Manifold with Vanishing Concircular Curvature Tensor and Quasi Concircular Curvature Tensor

The concircular curvature tensor $C(X,Y)Z$ is defined as

(3.1) $$C(X,Y)Z=R(X,Y)Z-\frac{r}{n(n-1)}\{g(Y,Z)X-g(X,Z)Y.$$

Putting $C=0$, we get

(3.2) $$R(X,Y)Z=\frac{r}{n(n-1)}\{g(Y,Z)X-g(X,Z)Y.$$

Taking $Z=\xi_1$ in (3.2) and using (2.4) and (2.12) (i) we get

$$\eta_1(Y)X-\eta_1(X)Y=\frac{r}{n(n-1)}\{\eta_1(Y)X-\eta_1(X)Y\}.$$

Taking $Y=\xi_1$ and using (2.1) we get

$$\eta_1(\xi_1)X-\eta_1(X)\xi_1=\frac{r}{n(n-1)}\{\eta_1(\xi_1)X-\eta_1(X)\xi_1\},$$

or $$-X-\eta_1(X)\xi_1=\frac{r}{n(n-1)}\{-X-\eta_1(X)\xi_1\},$$

or

(3.3) $$X[\frac{r}{n(n-1)}-1]=\eta_1(X)\xi_1[1-\frac{r}{n(n-1)}],$$

Contracting (3.3) we get

(3.4) $r = n(n-1)$, r the scalar curvature.

Using (3.4) in (3.2) we get

$$R(X,Y)Z = [g(Y,Z)X - g(X,Z)Y].$$

Thus a concircularly flat generalized Lorentzian Para-Sasakian manifold is of constant curvature and its value is +1. Hence a concircularly flat generalized Lorentzian Para-Sasakian manifold is locally isometric to a unit sphere $S^n(1)$. The quasi concircular curvature tensor $\tilde{C}$ is defined as

(3.5) $$\tilde{C}(X,Y)Z = aR(X,Y)Z - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\{g(Y,Z)X - g(X,Z)Y\},$$

where a,b are constants ,such that $ab \neq 0$.
Vanishing of $\tilde{C}$ implies that

(3.6) $$R(X,Y)Z = \frac{r}{an}\left(\frac{a}{n-1} + 2b\right)\{g(Y,Z)X - g(X,Z)Y\}.$$

Taking $Z = \xi_1$ in (3.6) and using (2.4) and (2.12)(i)

(3.7) $$\eta_1(Y)X - \eta_1(X)Y = \frac{r}{an}\left(\frac{a}{n-1} + 2b\right)\{\eta_1(Y)X - \eta_1(X)Y\}.$$

Taking $Y = \xi_1$ in (3.7) and using (2.1) we get

$$-X - \eta_1(X)\xi_1 = \frac{r}{an}\left(\frac{a}{n-1} + 2b\right)\{-X - \eta_1(X)\xi_1\},$$

(3.8) $$X\left[-1 + \frac{r}{an}\left(\frac{a}{n-1} + 2b\right)\right] = \eta_1(X)\xi_1\left[1 - \frac{r}{an}\left(\frac{a}{n-1} + 2b\right)\right],$$

Contracting (3.8) we get

(3.9) $$r = \frac{an}{\left(\frac{a}{n-1} + 2b\right)}.$$

Using (3.9) in (3.6) we get

$$R(X,Y)Z = [g(Y,Z)X - g(X,Z)Y].$$

Hence, a quasi concircularly flat Lorentzian Para-Sasakian manifold is locally isometric with a unit sphere $S^n(1)$.

4. Generalised Lorentzian Para-Sasakian Manifold Satisfying $R(X,Y)Z.C = 0$

Using (2.4) and (2.9)(i) in (3.1),we get

$$\eta_1(C(X,Y)Z) = [1 - \frac{r}{n(n-1)}][g(Y,Z)\eta_1(X) - g(X,Z)\eta_1(Y)]. \tag{4.1}$$

Putting $Z = \xi_1$ in (4.1) and using (2.4)

$$\eta_1(C(X,Y)\xi_1) = 0. \tag{4.2}$$

Now

$$\begin{aligned}(R(X,Y).C)(U,V)W &= R(X,Y)C(U,V)W - C(R(X,Y)U,V)W \\ &\quad - C(U,R(X,Y)V)W - C(U,V)R(X,Y)W.\end{aligned} \tag{4.3}$$

Using $R(X,Y).C = 0$ we get from (4.3)

$$\begin{aligned}&g[R(\xi_1,Y)C(U,V)W,\xi_1] - g[C(R(\xi_1,Y)U,V)W,\xi_1] \\ &\quad - g[C(U,R(\xi_1,Y)V)W,\xi_1] - g[C(U,V)R(\xi_1,Y)W,\xi_1] = 0.\end{aligned}$$

Using (2.4) and (2.10) (i), we get

$$\begin{aligned}&-C(U,V,W,Y) - \eta_1(Y)\eta_1(C(U,V)W) - g(Y,U)\eta_1(C(\xi_1,V)W) \\ &\quad + \eta_1(U)\eta_1(C(Y,V)W) - g(Y,V)\eta_1(C(U,\xi_1)W) + \eta_1(V)\eta_1(C(U,Y)W) \\ &\quad - g(Y,W)\eta_1(C(U,V)\xi_1) + \eta_1(W)\eta_1(C(U,V)Y) = 0,\end{aligned} \tag{4.4}$$

where $C(U,V,W,Y) = g(C(U,V)W,Y)$

Using (4.1) we get

(4.5) $$-\eta_1(Y)\eta_1(C(U,V)W)$$
$$=-\eta_1(Y)[1-\frac{r}{n(n-1)}][g(V,W)\eta_1(U)-g(U,W)\eta_1(V)],$$

(4.6) $$-g(Y,U)\eta_1(C(\xi_1,V)W)$$
$$=-g(Y,U)[1-\frac{r}{n(n-1)}][g(V,W)\eta_1(\xi_1)-g(\xi_1,W)\eta_1(V)],$$

(4.7) $$\eta_1(U)\eta_1(C(Y,V)W)$$
$$=\eta_1(U)[1-\frac{r}{n(n-1)}][g(V,W)\eta_1(Y)-g(Y,W)\eta_1(V)],$$

(4.8) $$-g(Y,V)\eta_1(C(U,\xi_1)W)$$
$$=-g(Y,V)[1-\frac{r}{n(n-1)}][g(\xi_1,W)\eta_1(U)-g(U,W)\eta_1(\xi_1)],$$

(4.9) $$\eta_1(V)\eta_1(C(U,Y)W)$$
$$=\eta_1(V)[1-\frac{r}{n(n-1)}][g(Y,W)\eta_1(U)-g(U,W)\eta_1(Y)],$$

(4.10) $$\eta_1(W)\eta_1(C(U,V)Y)$$
$$=\eta_1(W)[1-\frac{r}{n(n-1)}][g(V,Y)\eta_1(U)-g(U,Y)\eta_1(V)],$$

Using (4.5), (4.6), (4.7), (4.8), (4.9) and (4.10) in (4.4) we get,

(4.11) $$-C(U,V,W,Y)\ +[1-\frac{r}{n(n-1)}][g(Y,U)g(V,W)-g(Y,V)g(U,W)]=0.$$

Using (3.1) in (4.11) we get,

$$R(U,V,W,Y)=g(Y,U)g(V,W)-g(Y,V)g(U,W),$$

$$R(U,V)W=g(V,W)U-g(U,W)V.$$

Thus we have the following theorem

Theorem 1. *If in a generalized Lorentzian Para-Sasakian manifold* M *the relation* $R(X,Y).C = 0$ *holds, then it is locally isometric with a unit sphere* $S^n(1)$.

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