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Nonholonomic Frame for a Finsler Space with Special $(\alpha,\beta)-$ Metric

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Abstract: In the present paper, two Finsler deformations of the special $(\alpha,\beta)-$metric $F=\alpha+\varepsilon\frac{\beta^2}{\alpha}+k\frac{\beta^3}{\alpha^2}$, where ε and k are constants, of a Finsler space have been determined. The nonholonomic frame for the Finsler space with this special metric has been obtained. We also obtained some results which give nonholonomic frame for Finsler spaces with certain $(\alpha,\beta)-$metrics.

Key Words: Finsler space with $(\alpha,\beta)-$metric, Randers space, nonholonomic frame for a Finsler space, Riemannian metric, Finsler metric.

AMS Subject Classification: 53B40, 53C60.

1. Introduction

For a Finsler space with $(\alpha,\beta)-$ metric we have to deal with two metrics, the first one is original Riemannian metric a_{ij} and the second one is Finsler metric g_{ij}. In 1934, T. Hosokawa[1] discussed the nonholonomic system in a space of line-elements with an affine connection. The first Finsler space with $(\alpha,\beta)-$metric was introduced by G. Randers[2] in 1941 and it is known as Randers space. In 1951, Y. Katsurada[3] introduced the theory of nonholonomic system in a Finsler space. In 1982, P. R. Holland[4] ontained a new relation between the Finsler metric g_{ij} and the Riemannian metric a_{ij} for the particular case of a Randers space. In 1992, R. G. Beil[5] studied the gauge transformation viewed as a nonholonomic frame on a tangent bundle of a four dimensional base manifold. P. L. Antonelli[6, 7] found such nonholonomic frames for two important classes that are dual in the sense of Randers and Kropina spaces. Though the study of nonholonomic frames for Finsler spaces with $(\alpha,\beta)-$ metric is quite old and so many results have been obtained by several authors, still it is an important topic of research in Finsler geometry.

The aim of the present paper is to determine a nonholonomic frame for a Finsler space with a special $(\alpha,\beta)-$ metric $F=\alpha+\varepsilon\frac{\beta^2}{\alpha}+k\frac{\beta^3}{\alpha^2}$, where ε and k are constants.

2. Preliminaries

Definition 2.1: A Finsler space $F_n=(M, F(x,y))$ is called with $(\alpha,\beta)-$ metric, if there exists a 2-homogenous function L of the variables α and β such that the Finsler metric $F: TM\to\mathbb{R}$ is given by

$$F^2(x,y)=L(\alpha(x,y),\beta(x,y)) \tag{2.1}$$

where $\alpha^2(x,y)=a_{ij}(x)y^i y^j$, α is Riemannian metric on M and $\beta(x,y)=b_i(x)y^i$ is a 1-form on M.

Example 2.1: If $L(\alpha,\beta)=(\alpha+\beta)^2$, then the Finsler space with metric $F(x,y)=\sqrt{a_{ij}(x)y^i y^j}+b_i(x)y^i$ is called a Randers space.

Example 2.2: If $L(\alpha,\beta)=\frac{\alpha^4}{\beta^2}$, then the Finsler space with metric

$$F(x,y)=\frac{a_{ij}(x)y^i y^j}{|b_i(x)y^i|}$$

is called a Kropina space.

For a Finsler space with $(\alpha,\beta)-$metric $F^2(x,y)=L(\alpha,\beta)$, the following Finsler invariants are well known[8]

$$(2.2)\qquad \begin{cases} \rho=\dfrac{1}{2\alpha}\dfrac{\partial L}{\partial\alpha}, \\ \rho_0=\dfrac{1}{2}\dfrac{\partial^2 L}{\partial\beta^2}, \\ \rho_{-1}=\dfrac{1}{2\alpha}\dfrac{\partial^2 L}{\partial\alpha\partial\beta}, \\ \rho_{-2}=\dfrac{1}{2\alpha^2}\left(\dfrac{\partial^2 L}{\partial\alpha^2}-\dfrac{1}{\alpha}\dfrac{\partial L}{\partial\alpha}\right). \end{cases}$$

where the subscripts 1, 0, -1 and -2 indicate the degree of homogeneity of these invariants.

For a Finsler space with $(\alpha,\beta)-$metric, we have

$$(2.3)\qquad \rho_{-1}\beta+\rho_{-2}\alpha^2=0.$$

The metric tensor g_{ij} of a Finsler space with $(\alpha,\beta)-$metric is given by[9]

$$(2.4)\qquad g_{ij}(x,y) = \rho a_{ij}(x) + \rho_0 b_i(x)b_j(x) + \rho_{-1}\left(b_i(x)y_j + b_j(x)y_i\right) + \rho_{-2}y_iy_j.$$

From (2.4), we can see that g_{ij} is the result of two Finsler deformations

$$(2.5)\qquad a_{ij}\mapsto h_{ij}=\rho a_{ij}+\frac{1}{\rho_{-2}}\left(\rho_{-1}b_i+\rho_{-2}y_i\right)\left(\rho_{-1}b_j+\rho_{-2}y_j\right),$$

$$(2.6)\qquad h_{ij}\mapsto g_{ij}=h_{ij}+\frac{1}{\rho_{-2}}\left(\rho_0\rho_{-2}-\rho^2_{-1}\right)b_ib_j.$$

The nonholonomic Finsler frame that corresponds to the first deformation (2.5) is given by[10]

(2.7) $$X^{i}_{j}=\sqrt{\rho}\delta^{i}_{j}-\frac{1}{B^2}\left(\sqrt{\rho}\pm\sqrt{\rho+\frac{B^2}{\rho_{-2}}}\right)\left(\rho_{-1}b^{i}+\rho_{-2}y^{i}\right)\left(\rho_{-1}b_{j}+\rho_{-2}y_{j}\right),$$

where $B^2=a_{ij}\left(\rho_{-1}b^{i}+\rho_{-2}y^{i}\right)\left(\rho_{-1}b^{j}+\rho_{-2}y^{j}\right)=\rho_{-1}{}^{2}b^{2}+\beta\rho_{-1}\rho_{-2}$.

The metric tensors a_{ij} and h_{ij} are related as

(2.8) $$h_{ij}=X^{k}_{i}\,X^{l}_{j}a_{kl}\,.$$

The nonholonomic Finsler frame that corresponds to the second deformation (2.6) is given by[10]

(2.9) $$Y^{i}_{j}=\delta^{i}_{j}-\frac{1}{C^2}\left(1\pm\sqrt{1+\frac{\rho_{-2}C^2}{\rho_0\rho_{-2}-\rho^{2}_{-1}}}\right)b^{i}b_{j},$$

where $C^2=h_{ij}b^{i}b^{j}=\rho b^2+\frac{1}{\rho_{-2}}\left(\rho_{-1}b^2+\rho_{-2}\beta\right)^2$.

The metric tensors h_{ij} and g_{ij} are related as

(2.10) $$g_{ij}=\mathrm{Y}^{k}_{i}\,\mathrm{Y}^{l}_{j}\,h_{kl}\,.$$

From (2.8) and (2.10), we have

(2.11) $$g_{ij}=\mathrm{V}^{r}_{i}\,\mathrm{V}^{s}_{j}\,a_{rs}$$

where $\mathrm{V}^{r}_{i}=\mathrm{Y}^{k}_{i}\,\mathrm{X}^{r}_{k}$.

Theorem 2.1[11]: *Let $F^2(x,y)=L(\alpha,\beta)$ be a metric function of a Finsler space with $(\alpha,\beta)-$metric with condition (2.3). Then*

(2.12) $$\mathrm{V}^{i}_{j}=\mathrm{X}^{i}_{k}\,\mathrm{Y}^{k}_{j}$$

is a nonholonomic Finsler frame where X^{i}_{k} and Y^{k}_{j} are given by (2.7) and (2.9) respectively.

3. Nonholonomic Frame for a Finsler Space with Special $(\alpha,\beta)-$ Metric

Let us consider a Finsler space with special $(\alpha,\beta)-$metric

(3.1) $$F=\alpha+\varepsilon\frac{\beta^2}{\alpha}+k\frac{\beta^3}{\alpha^2},$$

where ε and k are constants.

Then the Finsler invariants (2.2) are given by

(3.2) $$\begin{cases}\rho=\dfrac{1}{\alpha^6}\left(\alpha^6-k\alpha^3\beta^3-\varepsilon^2\alpha^2\beta^4-3\varepsilon k\alpha\beta^5-2k^2\beta^6\right),\\[2ex] \rho_0=\dfrac{1}{\alpha^4}\left(2\varepsilon\alpha^4+6k\alpha^3\beta+6\varepsilon^2\alpha^2\beta^2+20\varepsilon k\alpha\beta^3+15k^2\beta^4\right),\\[2ex] \rho_{-1}=\dfrac{-\beta^2}{\alpha^6}\left(3k\alpha^3+4\varepsilon^2\alpha^2\beta+15\varepsilon k\alpha\beta^2+12k^2\beta^3\right),\\[2ex] \rho_{-2}=\dfrac{\beta^3}{\alpha^8}\left(3k\alpha^3+4\varepsilon^2\alpha^2\beta+15\varepsilon k\alpha\beta^2+12k^2\beta^3\right).\end{cases}$$

Also we have

(3.3) $$B^2=\frac{\beta^4}{\alpha^{14}}\left(3k\alpha^3+4\varepsilon^2\alpha^2\beta+15\varepsilon k\alpha\beta^2+12k^2\beta^3\right)^2\left(\alpha^2b^2-\beta^2\right)$$

and

(3.4) $$C^2=\frac{b^2}{\alpha^6}\left(\alpha^6-k\alpha^3\beta^3-\varepsilon^2\alpha^2\beta^4-3\varepsilon k\alpha\beta^5-2k^2\beta^6\right)+\frac{\beta}{\alpha^8}\left(3k\alpha^3+4\varepsilon^2\alpha^2\beta+15\varepsilon k\alpha\beta^2+12k^2\beta^3\right)\left(\alpha^2b^2-\beta^2\right)^2.$$

From (2.7) and (3.3), we have

$$X^i_j=\frac{1}{\alpha^3}\sqrt{\left(\alpha^6-k\alpha^3\beta^3-\varepsilon^2\alpha^2\beta^4-3\varepsilon k\alpha\beta^5-2k^2\beta^6\right)}\,\delta^i_j$$

(3.5)
$$-\frac{1}{\alpha}\left(\begin{array}{l}\sqrt{\left(\alpha^6-k\alpha^3\beta^3-\varepsilon^2\alpha^2\beta^4-3\varepsilon k\alpha\beta^5-2k^2\beta^6\right)}\\ \pm\sqrt{\left(\begin{array}{l}\alpha^6-4k\alpha^3\beta^3-5\varepsilon^2\alpha^2\beta^4-18\varepsilon k\alpha\beta^5\\ -14k^2\beta^6+3kb^2\alpha^5\beta+4\varepsilon^2b^2\alpha^4\beta^2\\ +15\varepsilon kb^2\alpha^3\beta^3+12k^2b^2\alpha^2\beta^4\end{array}\right)}\end{array}\right)\left(b^i-\frac{\beta}{\alpha^2}y^i\right)\left(b_j-\frac{\beta}{\alpha^2}y_j\right).$$

From (2.9), we have

(3.6)
$$Y^i_j=\delta^i_j-\frac{1}{C^2}\left(1\pm\sqrt{1+\frac{\alpha^4C^2}{2\varepsilon\alpha^4+3k\alpha^3\beta+2\varepsilon^2\alpha^2\beta^2+5\varepsilon k\alpha\beta^3+3k^2\beta^4}}\right)b^ib_j,$$

where C^2 is given by (3.4).

Thus, we have the following theorem.

Theorem 3.1: *Let $F^n=(M,F)$ be a Finsler space with special $(\alpha,\beta)-$ metric $F=\alpha+\varepsilon\frac{\beta^2}{\alpha}+k\frac{\beta^3}{\alpha^2}$ with condition (2.3). Then*

(3.7)
$$\mathrm{V}^i_j=\mathrm{X}^i_k\,\mathrm{Y}^k_j$$

is a nonholonomic Finsler frame where X^i_k and Y^k_j are given by (3.5) and (3.6) respectively.

We can find nonholonomic frames for certain Finsler spaces with $(\alpha,\beta)-$metric from theorem 3.1. There arises some cases.

Case (i): If $\varepsilon=1$ and $k=0$, we have $F=\alpha+\frac{\beta^2}{\alpha}$. In this case, Finsler invariants are

(3.8)
$$\begin{cases} \rho=\dfrac{1}{\alpha^4}\left(\alpha^4-\beta^4\right), \\ \rho_0=\dfrac{1}{\alpha^2}\left(2\varepsilon\alpha^2+6\varepsilon^2\beta^2\right), \\ \rho_{-1}=\dfrac{-4\varepsilon^2\beta^3}{\alpha^4}, \\ \rho_{-2}=\dfrac{4\varepsilon^2\alpha^2\beta^4}{\alpha^6}, \end{cases}$$

(3.9)
$$B^2=\frac{16\varepsilon^4\beta^6}{\alpha^{10}}\left(\alpha^2b^2-\beta^2\right)$$

and

(3.10)
$$C^2=\frac{b^2}{\alpha^4}\left(\alpha^4-\varepsilon^2\beta^4\right)+\frac{4\varepsilon^2\beta^2}{\alpha^6}\left(\alpha^2b^2-\beta^2\right)^2.$$

Hence the Finsler deformations of the Finsler metric are

(3.11)
$$X^i_{\ j}=\frac{1}{\alpha^3}\sqrt{\left(\alpha^6-\varepsilon^2\alpha^2\beta^4\right)}\,\delta^i_j - \frac{1}{\alpha}\left(\begin{matrix}\sqrt{\left(\alpha^6-\varepsilon^2\alpha^2\beta^4\right)} \\ \pm\sqrt{\left(\alpha^6-5\varepsilon^2\alpha^2\beta^4+4\varepsilon^2b^2\alpha^4\beta^2\right)}\end{matrix}\right)\left(b^i-\frac{\beta}{\alpha^2}y^i\right)\left(b_j-\frac{\beta}{\alpha^2}y_j\right).$$

and

(3.12)
$$Y^i_{\ j}=\delta^i_j-\frac{1}{C^2}\left(1\pm\sqrt{1+\frac{\alpha^4C^2}{2\varepsilon\alpha^4+2\varepsilon^2\alpha^2\beta^2}}\right)b^ib_j,$$

where C^2 is given by (3.10).

Thus we have

Corollary 3.1: *Let* $F^n=(M,F)$ *be a Finsler space with special* $(\alpha,\beta)-$ *metric* $F=\alpha+\dfrac{\beta^2}{\alpha}$ *with condition (2.3). Then*

$$\text{(3.13)} \qquad V^i_j = X^i_k\, Y^k_j$$

is a nonholonomic Finsler frame where X^i_k *and* Y^k_j *are given by (3.11) and (3.12) respectively.*

Case (ii): If $\varepsilon=0$ and $k=1$, we have $F=\alpha+\dfrac{\beta^3}{\alpha^2}$. In this case, Finsler invariants are

$$\text{(3.14)} \qquad \begin{cases} \rho=\dfrac{1}{\alpha^6}\left(\alpha^6-\alpha^3\beta^3-2\beta^6\right), \\ \rho_0=\dfrac{1}{\alpha^4}\left(6\alpha^3\beta+15\beta^4\right), \\ \rho_{-1}=\dfrac{-\beta^2}{\alpha^6}\left(3\alpha^3+12\beta^3\right), \\ \rho_{-2}=\dfrac{\beta^3}{\alpha^8}\left(3\alpha^3+12\beta^3\right), \end{cases}$$

$$\text{(3.15)} \qquad B^2=\frac{\beta^4}{\alpha^{14}}\left(3\alpha^3+12\beta^3\right)^2\left(\alpha^2b^2-\beta^2\right)$$

and

$$\text{(3.16)} \qquad C^2=\frac{b^2}{\alpha^6}\left(\alpha^6-\alpha^3\beta^3-2\beta^6\right)+\frac{\beta}{\alpha^8}\left(3\alpha^3+12\beta^3\right)\left(\alpha^2b^2-\beta^2\right)^2.$$

Hence the Finsler deformations of the Finsler metric are

$$X^i_j = \frac{1}{\alpha^3}\sqrt{\left(\alpha^6-\alpha^3\beta^3-2\beta^6\right)}\,\delta^i_j$$

(3.17)
$$-\frac{1}{\alpha}\left(\begin{array}{c}\sqrt{\left(\alpha^6-\alpha^3\beta^3-2\beta^6\right)} \\ \pm\sqrt{\left(\begin{array}{c}\alpha^6-4\alpha^3\beta^3-14\beta^6 \\ +3b^2\alpha^5\beta+12b^2\alpha^2\beta^4\end{array}\right)}\end{array}\right)\left(b^i-\frac{\beta}{\alpha^2}y^i\right)\left(b_j-\frac{\beta}{\alpha^2}y_j\right).$$

and

(3.18)
$$Y^i_j=\delta^i_j-\frac{1}{C^2}\left(1\pm\sqrt{1+\frac{\alpha^4C^2}{3\alpha^3\beta+3\beta^4}}\right)b^i b_j,$$

where C^2 is given by (3.16).

Thus we have

Corollary 3.2: *Let* $F^n=(M,F)$ *be a Finsler space with special* $(\alpha,\beta)-$ *metric* $F=\alpha+\dfrac{\beta^3}{\alpha^2}$ *with condition (2.3). Then*

(3.19)
$$V^i_j = X^i_k\, Y^k_j$$

is a nonholonomic Finsler frame where X^i_k *and* Y^k_j *are given by (3.17) and (3.18) respectively.*

Case (iii): If $\varepsilon=1$ and $k=1$, we have $F=\alpha+\dfrac{\beta^2}{\alpha}+\dfrac{\beta^3}{\alpha^2}$.

In this case, Finsler invariants are

$$\text{(3.20)} \quad \begin{cases} \rho = \dfrac{1}{\alpha^6}\left(\alpha^6 - \alpha^3\beta^3 - \alpha^2\beta^4 - 3\alpha\beta^5 - 2\beta^6\right), \\ \rho_0 = \dfrac{1}{\alpha^4}\left(2\alpha^4 + 6\alpha^3\beta + 6\alpha^2\beta^2 + 20\alpha\beta^3 + 15\beta^4\right), \\ \rho_{-1} = \dfrac{-\beta^2}{\alpha^6}\left(3\alpha^3 + 4\alpha^2\beta + 15\alpha\beta^2 + 12\beta^3\right), \\ \rho_{-2} = \dfrac{\beta^3}{\alpha^8}\left(3\alpha^3 + 4\alpha^2\beta + 15\alpha\beta^2 + 12\beta^3\right), \end{cases}$$

$$\text{(3.21)} \quad B^2 = \frac{\beta^4}{\alpha^{14}}\left(3\alpha^3 + 4\alpha^2\beta + 15\alpha\beta^2 + 12\beta^3\right)^2\left(\alpha^2 b^2 - \beta^2\right)$$

and

$$\text{(3.22)} \quad \begin{aligned} C^2 = {} & \frac{b^2}{\alpha^6}\left(\alpha^6 - \alpha^3\beta^3 - \alpha^2\beta^4 - 3\alpha\beta^5 - 2\beta^6\right) \\ & + \frac{\beta}{\alpha^8}\left(3\alpha^3 + 4\alpha^2\beta + 15\alpha\beta^2 + 12\beta^3\right)\left(\alpha^2 b^2 - \beta^2\right)^2. \end{aligned}$$

Hence the Finsler deformations of the Finsler metric are

$$\text{(3.23)} \quad \begin{aligned} X^i_j = {} & \frac{1}{\alpha^3}\sqrt{\left(\alpha^6 - \alpha^3\beta^3 - \alpha^2\beta^4 - 3\alpha\beta^5 - 2\beta^6\right)}\,\delta^i_j \\ & - \frac{1}{\alpha}\left(\begin{array}{l} \sqrt{\left(\alpha^6 - \alpha^3\beta^3 - \alpha^2\beta^4 - 3\alpha\beta^5 - 2\beta^6\right)} \\ \pm\sqrt{\left(\begin{array}{l} \alpha^6 - 4\alpha^3\beta^3 - 5\alpha^2\beta^4 - 18\alpha\beta^5 \\ -14\beta^6 + 3b^2\alpha^5\beta + 4b^2\alpha^4\beta^2 \\ +15b^2\alpha^3\beta^3 + 12b^2\alpha^2\beta^4 \end{array}\right)} \end{array}\right)\left(b^i - \frac{\beta}{\alpha^2}y^i\right)\left(b_j - \frac{\beta}{\alpha^2}y_j\right). \end{aligned}$$

and

$$\text{(3.24)} \quad Y^i_j = \delta^i_j - \frac{1}{C^2}\left(1 \pm \sqrt{1 + \frac{\alpha^4 C^2}{2\alpha^4 + 3\alpha^3\beta + 2\alpha^2\beta^2 + 5\alpha\beta^3 + 3\beta^4}}\right) b^i b_j,$$

where C^2 is given by (3.22).

Thus we have

Corollary 3.3: *Let* $F^n=(M,F)$ *be a Finsler space with special* $(\alpha,\beta)-$ *metric* $F=\alpha+\frac{\beta^2}{\alpha}+\frac{\beta^3}{\alpha^2}$ *with condition (2.3). Then*

(3.25) $$V^i_j = X^i_k\, Y^k_j$$

is a nonholonomic Finsler frame where X^i_k *and* Y^k_j *are given by (3.23) and (3.24) respectively.*

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