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Study of a Semi-Symmetric Space with a Non-Recurrent Torsion Tensor

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Abstract: In this paper, we have generalized the condition given by Yilmaze, Zengin and Uysal[1] for torsion tensor in the study of a Riemannian manifold equipped with semi-symmetric metric connection.

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1. Introduction

Let $(M^n, g)(\mathrm{n} > 2)$, be an n-dimensional Riemannian manifold with Riemannian metric g. A connection ∇ is said to be symmetric connection if torsion tensor T of the connection vanishes otherwise it is called a non-symmetric connection. In 1932, H. A. Hayden[2] introduced the idea of semi-symmetric connection. A connection ∇ is said to be a metric connection if

(1.1) $$\nabla g = 0,$$

otherwise it is called a non-metric connection. The Riemannian manifold equipped with a semi-symmetric metric connection has been studied by O. C. Andonie[3] , M. C. Chaki and A. Konar[4], B. B. Chaturvedi and P. N. Pandey[5], P. N. Pandey and B. B. Chaturvedi[6, 7] and B. B. Chaturvedi and B. K. Gupta[8, 9].

A. Friedman and J. A. Schouten[10] considered a semi-symmetric metric connection ∇ and Riemannian connection D associated with coefficients Γ_{ij}^h and $\left\{ {h \atop ij} \right\}$ respectively. According to them if the torsion tensor T of the connection ∇ on $(M^n, g)(\mathrm{n} > 2)$, satisfies

(1.2) $$T_{ij}^h = \delta_i^h \omega_j - \delta_j^h \omega_i .$$

Then

(1.3) $$\Gamma_{ij}^h = \left\{ {h \atop ij} \right\} + \delta_i^h \omega_j - g_{ij} \omega^h ,$$

where ω^h are contravariant components of the generating vector ω_h such that $\omega^h = g^{th} \omega_t$ and

(1.4) $$\nabla_j \omega_i = D_j \omega_i - \omega_i \omega_j + g_{ij} \omega ,$$

where $\omega = \omega^h \omega_h$.

A. Friedman and J. A. Schouten[10] obtained the relation between curvature tensor with respect to semi- symmetric metric connection and Riemannian connection given by

(1.5) $$\bar{R}_{ijkm} = R_{ijkm} - g_{im} \pi_{jk} + g_{jm} \pi_{ik} - g_{jk} \pi_{im} + g_{ik} \pi_{jm} ,$$

where

(1.6) $$\pi_{jk} = \nabla_j \omega_k - \omega_j \omega_k + \frac{1}{2} g_{jk} \omega.$$

Transvecting (1.5) by g^{im}, we get

(1.7) $$\bar{R}_{jk} = R_{jk} - (n-2)\pi_{jk} - g_{jk} \pi.$$

Transvecting (1.7) by g^{jk}, we get

(1.8) $$\bar{R} = R - 2(n-1)\pi.$$

2. Non-Recurrent Torsion Tensor

In 2001, U. C. De and J. Sengupta[11] studied some properties of an almost contact manifold equipped with semi-symmetric metric connection whose torsion tensor satisfies a special condition. In the continuation of these developments H. B. Yilmaze, F. O. Zengin and S. A. Uysal[1] studied a semi-symmetric metric connection with a special condition on a Riemannian manifold. They considered a Riemannian manifold equipped with a semi-symmetric metric connection whose torsion tensor T satisfies the following condition

(2.1) $$\nabla_j T^h_{ik} = a_j T^h_{ik} + b_j b^h g_{ik} + \delta^h_j b_i a_k \; .$$

Above developments motivated us to study a semi-symmetric space equipped with a non-recurrent torsion tensor.

We define a non-recurrent torsion tensor by generalizing the expression of torsion tensor given by H. B. Yilmaze, F. O. Zengin and S. A. Uysal[1]

(2.2) $$\nabla_j T^h_{ik} = a_j T^h_{ik} + b_j b^h g_{ik} + b^h b_k g_{ij} + b_i b^h g_{jk} \, ,$$

where $b_k = b^l g_{lk}, b^l = b_k g^{kl}$ and a_j, b_j are non-zero orthogonal vector fields.

Contracting the indices h and i in (1.2), we get

(2.3) $$T^h_{hj} = (n-1)\omega_j \, .$$

Taking covariant derivative with respect to semi-symmetric metric connection ∇ of (2.3), we have

(2.4) $$\nabla_k T^h_{hj} = (n-1)\nabla_k \omega_j \, .$$

Contracting the indices h and i in (2.2) and using $b_k = b^l g_{lk}$ and $b = b^l b_l$, we get

(2.5) $$\nabla_j T^h_{hk} = a_j T^h_{hk} + 2 b_j b_k + b g_{jk} \, ,$$

From (2.3), (2.4) and (2.5), we get

(2.6) $$\nabla_j \omega_k = a_j \omega_k + \frac{2}{n-1} b_j b_k + \frac{b g_{jk}}{n-1}.$$

From (1.6) and (2.6), we get

(2.7) $$\pi_{jk} = a_j \omega_k + \frac{2}{n-1} b_j b_k + \frac{b g_{jk}}{n-1} - \omega_j \omega_k + \frac{1}{2} g_{jk} \omega .$$

Defining

(2.8) $$\alpha_{jk} = \frac{2}{n-1} b_j b_k + \frac{b g_{jk}}{n-1} - \omega_j \omega_k + \frac{1}{2} g_{jk} \omega \text{ and } \alpha = \alpha_{ij} g^{ij},$$

and using (2.8) in (2.7), we have

(2.9) $$\pi_{jk} = a_j \omega_k + \alpha_{jk} .$$

Using (2.9) in (1.5), we get

(2.10) $$\begin{aligned}\bar{R}_{ijkh} = R_{ijkh} - g_{ih}\alpha_{jk} + g_{jh}\alpha_{ik} - g_{jk}\alpha_{ih} + g_{ik}\alpha_{jh} \\ - g_{ih} a_j \omega_k + g_{jh} a_i \omega_k - g_{jk} a_i \omega_h + g_{ik} a_j \omega_h .\end{aligned}$$

Transvecting (2.10) with g^{kh}, we find

(2.11) $$\bar{R}_{ij} = R_{ij} .$$

Thus we conclude:

Theorem 2.1: If *the torsion tensor of a Riemannian manifold* (M^n, g) (n >2) *equipped with a semi symmetric metric connection is non-recurrent with respect to the semi-symmetric metric connection (i.e. it satisfies (2.2)), then*

$$\bar{R}_{ij} = R_{ij},$$

where $\bar{R}_{ij}$ *and* R_{ij} *are Ricci tensor with respect to the semi-symmetric metric connection and the Riemannian connection respectively.*

Now we propose:

Theorem 2.2: If *the torsion tensor of a Riemannian manifold* (M^n, g) (n >2) *equipped with a semi symmetric metric connection is non-recurrent with respect to the semi-symmetric metric connection (i.e. it satisfies (2.2)), then the curvature tensor of type (0,4) with respect to semi-symmetric metric connection satisfies the following*

1. $\bar{R}_{ijkh} = -\bar{R}_{jikh}$ *i.e. skew symmetric in first two indices,*

2. $\bar{R}_{ijkh} + \bar{R}_{jkih} + \bar{R}_{kijh} = 0$, *if* $a_j\omega_k = a_k\omega_j$.

Proof: Interchanging the indices i and j in (2.10), we get

$$\bar{R}_{jikh} = R_{jikh} - g_{jh}\alpha_{ik} + g_{ih}\alpha_{jk} - g_{ik}\alpha_{jh} + g_{jk}\alpha_{ih} - g_{jh}a_i\omega_k + g_{ih}a_j\omega_k - g_{ik}a_j\omega_h + g_{jk}a_i\omega_h . \tag{2.12}$$

Adding (2.10) and (2.12), we get

$$\bar{R}_{ijkh} + \bar{R}_{jikh} = R_{ijkh} + R_{jikh} . \tag{2.13}$$

Since in Riemannian manifold the curvature tensor satisfies

$$R_{ijkh} + R_{jikh} = 0 . \tag{2.14}$$

Hence from (2.13) and (2.14) we get (i).

Now interchanging i, j and k cyclically in (2.10), we get

$$\bar{R}_{ijkh} = R_{ijkh} - g_{ih}\alpha_{jk} + g_{jh}\alpha_{ik} - g_{jk}\alpha_{ih} + g_{ik}\alpha_{jh} - g_{ih}a_j\omega_k + g_{jh}a_i\omega_k - g_{jk}a_i\omega_h + g_{ik}a_j\omega_h , \tag{2.15}$$

$$\bar{R}_{jkih} = R_{jkih} - g_{jh}\alpha_{ki} + g_{kh}\alpha_{ji} - g_{ik}\alpha_{jh} + g_{ij}\alpha_{kh} - g_{jh}a_k\omega_i + g_{kh}a_j\omega_i - g_{ik}a_j\omega_h + g_{ij}a_k\omega_h , \tag{2.16}$$

and

(2.17) $$\bar{R}_{kijh} = R_{kijh} - g_{kh}\alpha_{ij} + g_{ih}\alpha_{kj} - g_{ij}\alpha_{kh} + g_{kj}\alpha_{ih}$$
$$- g_{kh}a_i\omega_j + g_{ih}a_k\omega_j - g_{ji}a_k\omega_h + g_{jk}a_i\omega_h .$$

Adding (2.15), (2.16) and (2.17), we have

(2.18) $$\bar{R}_{ijkh} + \bar{R}_{jkih} + \bar{R}_{kijh} = R_{ijkh} + R_{jkih} + R_{kijh} + g_{jh}(a_i\omega_k - a_k\omega_i)$$
$$+ g_{ih}(a_k\omega_j - \omega_k a_j) + g_{kh}(a_j\omega_i - \omega_j a_i) .$$

From (2.18), we get

(2.19) $$\bar{R}_{ijkh} + \bar{R}_{jkih} + \bar{R}_{kijh} = R_{ijkh} + R_{jkih} + R_{kijh} \text{ if } a_j\omega_k = a_k\omega_j .$$

Since in Riemannian manifold the curvature tensor satisfies the relation

(2.20) $$R_{ijkh} + R_{jkih} + R_{kijh} = 0 .$$

Hence from (2.19) and (2.20), we get (ii).

Now we propose:

Theorem 2.3: If *the torsion tensor of a Riemannian manifold* (M^n, g) (n >2) *equipped with a semi symmetric metric connection is non-recurrent with respect to the semi-symmetric metric connection (i.e. it satisfies (2.2)), then the scalar curvature with respect to the semi-symmetric metric connection has the form*

$$\bar{R} = R - 2(n-1)\alpha - 2(n-1)a_p\omega^p .$$

Proof: Now transvecting (2.10) with g^{ih}, we get

(2.21) $$\bar{R}_{jk} = R_{jk} - (n-2)\alpha_{jk} - (n-2)a_j\omega_k - \alpha g_{jk} - a_p\omega^p g_{jk} .$$

Transvecting (2.21) with g^{jk} and using $g^{jk}g_{jk} = n$, we get

(2.22) $$\bar{R} = R - 2(n-1)\alpha - 2(n-1)a_p\omega^p .$$

Now we propose:

Corollary 2.1: If *the torsion tensor of a Riemannian manifold* (M^n, g) (n >2) *equipped with a semi symmetric metric connection is non-recurrent with respect to the semi-symmetric metric connection (i.e. it satisfies (2.2)), then the scalar curvature with respect to the semi-symmetric metric connection is equal to Riemannian scalar curvature if and only if*

$$\alpha = -a_p \omega^p .$$

Proof: If we take $\overline{R} = R$ then from equation (2.22), we have

(2.23) $$2(n-1)(\alpha + a_p \omega^p) = 0 .$$

Since $n \neq 1$ then from (2.23), we get

(2.24) $$\alpha = -a_p \omega^p .$$

Let $\alpha = -a_p \omega^p$ and using in (2.22), we get

(2.25) $$\overline{R} = R .$$

3. Conharmonic Curvature Tensor

Definition 3.1: The Conharmonic curvature tensor L of type (0, 4) in a Riemannian manifold is defined as

(3.1) $$L_{ijkh} = R_{ijkh} - \frac{1}{n-2}(g_{ih} R_{jk} - g_{ik} R_{jh} + g_{jk} R_{ih} - g_{jh} R_{ik}) .$$

The Conharmonic curvature tensor with respect to semi-symmetric metric connection is given by

(3.2) $$\overline{L}_{ijkh} = \overline{R}_{ijkh} - \frac{1}{n-2}(g_{ih} \overline{R}_{jk} - g_{ik} \overline{R}_{jh} + g_{jk} \overline{R}_{ih} - g_{jh} \overline{R}_{ik}) ,$$

using (2.10) and (2.21) in (3.2), we get

$$\bar{L}_{ijkh} = R_{ijkh} - g_{ih}\alpha_{jk} + g_{jh}\alpha_{ik} - g_{jk}\alpha_{ih} + g_{ik}\alpha_{jh}$$
$$- g_{ih}a_j\omega_k + g_{jh}a_i\omega_k - g_{jk}a_i\omega_h + g_{ik}a_j\omega_h \tag{3.3}$$
$$-\frac{1}{n-2}\begin{bmatrix} g_{ih}(R_{jk} - (n-2)\alpha_{jk} - (n-2)a_j\omega_k - \alpha g_{jk} - a_p\omega^p g_{jk}) \\ -g_{ik}(R_{jh} - (n-2)\alpha_{jh} - (n-2)a_j\omega_h - \alpha g_{jh} - a_p\omega^p g_{jh}) \\ +g_{jk}(R_{ih} - (n-2)\alpha_{ih} - (n-2)a_i\omega_h - \alpha g_{ih} - a_p\omega^p g_{ih}) \\ -g_{jh}(R_{ik} - (n-2)\alpha_{ik} - (n-2)a_i\omega_k - \alpha g_{ik} - a_p\omega^p g_{ik}) \end{bmatrix}.$$

Equation (3.3) implies

$$\bar{L}_{ijkh} = R_{ijkh} - \frac{1}{n-2}(g_{ih}R_{jk} - g_{ik}R_{jh} + g_{jk}R_{ih} - g_{jh}R_{ik}) \tag{3.4}$$
$$-\frac{2}{n-2}(\alpha + a_p\omega^p)(g_{jh}g_{ik} - g_{jk}g_{ih}).$$

Using (3.1) in (3.4), we have

$$\bar{L}_{ijkh} = L_{ijkh} - \frac{2}{n-2}(\alpha + a_p\omega^p)(g_{jh}g_{ik} - g_{jk}g_{ih}). \tag{3.5}$$

Now from (3.5), we can say that $\bar{L}_{ijkh} = L_{ijkh}$ if and only if $\alpha = -a_p\omega^p$ or $g_{jh}g_{ik}$ be symmetric in i and j or $g_{jh}g_{ik}$ be symmetric in k and h.

Thus we conclude:

Theorem 3.1: If *the torsion tensor of a Riemannian manifold* (M^n, g) (n >2) *equipped with a semi symmetric metric connection is non-recurrent with respect to the semi-symmetric metric connection (i.e. it satisfies (2.2)), then the Conharmonic curvature tensor with respect to Riemannian connection is equal to Conharmonic curvature tensor with respect to semi-symmetric metric connection if and only if at least one of the following holds*

i. $\alpha = -a_p\omega^p$,

ii. $g_{jh}g_{ik}$ *be symmetric in* i *and* j.

iii. $g_{jh}g_{ik}$ *be symmetric in* h *and* k.

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