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Certain F_q-Functions Associated with Ramanujan' Mock Theta Functions-I

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Abstract: This paper is in continuation of my earlier papers[1, 2] in which we have defined and studied the q-differential-difference equation

$$D_{q,z}F(z,\alpha)=F(z,\alpha+1),$$

where $$zD_{q,z}F(z,\alpha)=F(z,\alpha)-F(zq,\alpha).$$

In this paper, we have given certain generalized Mock Theta Functions of orders three, five seven which satisfy the above equation and reduce to the original Mock Theta Functions. Some of the common properties have also been studied.

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1. Introduction

In papers[1,2], we have discussed the properties of the class of functions which satisfy the q- differential-difference equation

(1.1) $$D_{q,z}F(z,\alpha)=F(z,\alpha+1),$$

where $zD_{q,z}F(z,\alpha)=F(z,\alpha)-F(zq,\alpha)$.

We call (1.1) as the F_q-equation and functions which satisfy this equation, as F_q-functions.

A very important class of functions discovered by Ramanujan is the famous Mock Theta Functions. There are 27 of them known to us-seven of three, ten of order five, three of order seven and seven more which have been found recently in the "Lost" Notebook and studied by Andrews and Hickerson[3].

In this paper we define certain generalized functions which reduce to each of these Mock Theta Functions and which also satisfy the F_q-equation

(1.1). In subsequent sections some of the common properties of these generalized functions have been studied.

The most revealing property is the q-differential property of the generalized functions associated with mock theta functions of order five and seven.

We shall have occasion to use the q-analogues of the trigonometrical functions, defined as

$$Sin_q(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(q)_{2n+1}}; \qquad |x|<1,$$

$$Cos_q(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(q)_{2n}}; \qquad |x|<1.$$

The third order Mock Theta Functions given by Ramanujan and studied by Watson[4] and redefined in terms of ${}_2\Phi_1$-series by Fine[5] and Agarwal[6] are as follows:

(1.2)

$$(i) \quad f(q)=2-{}_2\Phi_1\left(q,0;-q;-q\right),$$

$$(ii) \quad \Phi(q)=(1+i)-i\,{}_2\Phi_1\left(q,0;-iq;\, iq\right),$$

$$(iii) \quad \Psi(q)=q\,{}_2\Phi_1\left(q^2\right)\left(q^2,-q^2;0;\, q\right),$$

$$(iv) \quad \chi(q)=(1+\omega)-\omega\,{}_2\Phi_1\left(0,q;-\omega q;\,-\omega^2 q\right),$$

$$(v) \quad \omega(q)=(1-q)^{-1}{}_2\Phi_1\left(q^2\right)\left(q^2,0;q^3;\, q\right),$$

$$(vi) \quad \nu(q)={}_2\Phi_1\left(q^2\right)\left(q^2,q;0;\,-q\right),$$

$$(vii) \quad \rho(q)=(1-\omega q)^{-1}{}_2\Phi_1\left(q^2\right)\left(q^2,0;\omega q^3;\, q\omega^{-1}\right),$$

where $\omega=e^{2\pi i/3}$.

The ten fifth order mock theta functions are

$$(i) \quad f_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q)_n},$$

$$(ii) \quad \Phi_0(q)=\sum_{n=0}^{\infty}q^{n^2}\left(-q;\, q^2\right)_n,$$

$$(iii) \quad \Psi_0(q)=\sum_{n=0}^{\infty} q^{(n+1)(n+2)/2}\left(-q\right)_n,$$

$$(iv) \quad F_0(q)=\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;-q^2)_n},$$

$$(v) \quad \chi_0(q)=\sum_{n=0}^{\infty}\frac{q^n}{(q^{n+1})_n},$$

$$(vi) \quad f_1(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(-q)_n},$$

$$(vii) \quad \Phi_1(q)=\sum_{n=0}^{\infty} q^{(n+1)^2}\left(-q;\, q^2\right)_n,$$

$$(viii) \quad \Psi_1(q)=\sum_{n=0}^{\infty} q^{n(n+1)/2}\left(-q\right)_n,$$

$$(ix) \quad F_1(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;\, q^2)_{n+1}},$$

$$(1.3) \qquad (x) \quad \chi_1(q)=\sum_{n=0}^{\infty}\frac{q^n}{(q^{n+1})_{n+1}}.$$

The three seventh order mock theta functions are

$$(1.4) \qquad (i) \quad \Im_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^{n+1})_n}, \quad (ii) \quad \Im_1(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^n)_n},$$

$$(iii) \quad \Im_2(q)=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q^{n+1})_{n+1}}.$$

2. Certain F_q-Functions Associated with Mock Theta Functions of Order Three

Let us define a function $f\left(x,\alpha\right)$ as

$$\text{(i)} \qquad f\left(x,\alpha\right)=\frac{(q)_\infty}{(x)_\infty}\sum_{n=0}^{\infty}\frac{(x)_n}{\left(q,-q\right)_n}\left(-q^{\alpha}\right)^n, \qquad |q|<1,\ \mathrm{Re}(\alpha)>0.$$

Now $D_{q,x} f(x,\alpha) = \frac{(q)_\infty}{(x)_\infty} \sum_{n=0}^{\infty} \frac{(x)_n}{(q,-q)_n} \left(-q^{\alpha+1}\right)^n = f(x,\alpha+1).$

Hence $f(x,\alpha)$ is a F_q-function.

Similarly, the other generalized F_q-functions associated with the third order mock theta functions can be listed as below: for $|q|<1$, $\mathrm{Re}(\alpha)>0$,

$$(ii) \quad \Phi(x,\alpha) = (q)_\infty (x)_\infty^{-1} \sum_{n=0}^{\infty} \frac{(x)_n}{(q,-iq)_n} \left(-iq^{\alpha}\right)^n,$$

$$(iii) \quad \Psi(x,\alpha) = \frac{q(q)_\infty}{(x)_\infty} \sum_{n=0}^{\infty} \frac{(iq,-iq,\ x)_n}{(q)_n} q^{\alpha n},$$

$$(iv) \quad \chi(x,\alpha) = (1+\omega^2) \frac{(q)_\infty}{(x)_\infty} \sum_{n=0}^{\infty} \frac{(x)_n}{(q,-\omega q)_n} \left(-\omega^2 q^{\alpha}\right)^n,$$

$$(v) \quad \omega(x,\alpha) = (1-q)^{-1} \frac{(q)_\infty}{(x)_\infty} \sum_{n=0}^{\infty} \frac{(x)_n\, q^{\alpha n}}{\left(q, q\sqrt{q}\right)_n \left(-q\sqrt{q}\right)_n},$$

$$(vi) \quad \nu(x,\alpha) = (q)_\infty (x)_\infty^{-1} \sum_{n=0}^{\infty} \frac{\left(\sqrt{q},-\sqrt{q},x\right)_n}{(q)_n} \left(-q^{\alpha}\right)^n,$$

(2.1)

$$(vii) \quad \rho(x,\alpha) = \frac{(q)_\infty}{(x)_\infty (1-\omega q)} \sum_{n=0}^{\infty} \frac{(x)_n \left(\omega^{-1} q^{\alpha}\right)^n}{\left(q, q\sqrt{\omega q}\right)_n \left(-q\sqrt{\omega q}\right)_n}.$$

For $x=q$ and $\alpha=1$ all the above functions (2.1(i-vii)) reduces to the corresponding third order mock theta functions [2-f(q)], $\left[(1-i)+i\Phi(q)\right]$, $\Psi(q)$, $\omega^2+\chi(q)$, $\omega(q)$, $\nu(q)$ and $\rho(q)$, defined in (1.2)(i-vii).

It is interesting to note that if we take values of x and α other than q and 1 respectively, in (2.1(i)-(vii)), we get the following known functions

$$(i)\quad f\left(-q,\alpha\right)=(q)_\infty e_q(-q)e_q(-q^\alpha)$$
$$=(q)_\infty\sum_{n=-\infty}^{\infty}\left(iq^{\frac{1}{2}(1-\alpha)}\right)^n {}_q j_n\left(2q^{\frac{1}{2}(1+\alpha)}i\right),$$

$$(ii)\quad \Phi\left(-iq,\alpha\right)=(q)_\infty e_q(-iq)e_q(-iq^\alpha)$$
$$=(q)_\infty\sum_{n=-\infty}^{\infty}\left((-)q^{\frac{1}{2}(1-\alpha)}\right)^n {}_q j_n\left(2q^{\frac{1}{2}(1+\alpha)}i\right),$$

$$(iii)\quad \Psi(q,\alpha)=q_2\Phi_1^{(q^2)}\left(-q^2,q^2;0;\,q^\alpha\right),$$

$$(iv)\quad \chi\left(-\omega q,\alpha\right)=(q)_\infty\left(1+\omega^2\right)e_q(-\omega q)e_q(-\omega^2 q^\alpha)$$
$$=(q)_\infty\left(1+\omega^2\right)\sum_{n=-\infty}^{\infty}\left(i\omega q^{\frac{1}{2}(1-\alpha)}\right)^n {}_q j_n\left(2q^{\frac{1}{2}(1+\alpha)}i\right),$$

$$(v)\quad \omega(q,\alpha)=(1-q)^{-1}{}_2\Phi_1^{(q^2)}\left(q^2,0;q^3;\,q^\alpha\right),$$

(2.2) $$(vi)\quad \nu(q,\alpha)={}_2\Phi_1^{(q^2)}\left(q^2,q;0;\,-q^\alpha\right),$$

and $$(vii)\quad \rho(q,\alpha)=(1-\omega q)^{-1}{}_2\Phi_1^{(q^2)}\left(q^2,0;\omega q^3;\,\omega^{-1}q^\alpha\right),$$

where ${}_q j_\nu(x)$ is a q-Bessel function defined in chapter 2, and satisfies the generating function (see[6])

$$\sum_{-\infty}^{\infty} t^n\,{}_q j_n(z)=e_q(zt/2)e_q(z/2t).$$

3. Certain q-integral Representing for the Functions (2.1(i)-(vii))

Using the following limiting case of a known integral representation for the q-Beta function, namely

(3.1)
$$\frac{1}{(q^x;q)_\infty} = \frac{(1-q)^{-1}}{(q)_\infty}\int_0^1 t^{x-1}(tq;q)_\infty \, d_q t, \quad \operatorname{Re} x > 0.$$

We can write the following q-integral representations $(q^\alpha \equiv a)$ of the function defined in (2.1(i)-(vii)) valid for $\operatorname{Re} x > 0, |\alpha| < 1;$

(3.2)

$$(i) \quad (q)_\infty f(q^x, a) = (1-q)^{-1}\int_0^1 (tq)_\infty f(0, at)\, d_q t,$$

$$(ii) \quad (q)_\infty \Phi(q^x, a) = (1-q)^{-1}\int_0^1 t^{x-1}(tq)_\infty \Phi(0, at)\, d_q t,$$

$$(iii) \quad (q)_\infty \Psi(q^x, a) = (1-q)^{-1}\int_0^1 t^{x-1}(tq)_\infty \Psi(0, at)\, d_q t,$$

$$(iv) \quad (q)_\infty \chi(q^x, a) = (1-q)^{-1}\int_0^1 t^{x-1}(tq)_\infty \chi(0, at)\, d_q t,$$

$$(v) \quad (q)_\infty \omega(q^x, a) = (1-q)^{-1}\int_0^1 t^{x-1}(tq)_\infty \omega(0, at)\, d_q t,$$

$$(vi) \quad (q)_\infty \nu(q^x, a) = (1-q)^{-1}\int_0^1 t^{x-1}(tq)_\infty \nu(0, at)\, d_q t,$$

$$(vii) \quad (q)_\infty \rho(q^x, a) = (1-q)^{-1}\int_0^1 t^{x-1}(tq)_\infty \rho(0, at)\, d_q t,$$

If we substitute *x=1* and *a=q* in the above q-integrals, we get the corresponding q-integral representations for [2-f(q)], $[(1-i)+i\Phi(q)]$, $\Psi(q)$, $\omega^2+\chi(q)$, $\omega(q)$, $\nu(q)$ and $\rho(q)$, respectively, namely

$$(i)\quad (q)_\infty\left[2-\mathrm{f}(\mathrm{q})\right]=(1-q)^{-1}\int_0^1 t^{x-1}(tq)_\infty f(0,\,at)\,d_qt,$$

$$(ii)\quad (q)_\infty\left[(1-i)+i\Phi(q)\right],=(1-q)^{-1}\int_0^1 (tq)_\infty \Phi(0,\,at)\,d_qt,$$

$$(iii)\quad (q)_\infty \Psi(q)=(1-q)^{-1}\int_0^1 (tq)_\infty \Psi(0,\,at)\,d_qt,$$

$$(iv)\quad (q)_\infty\left[\omega^2+\chi(q)\right]=(1-q)^{-1}\int_0^1 (tq)_\infty \chi(0,\,at)\,d_qt,$$

$$(v)\quad (q)_\infty \omega(q)=(1-q)^{-1}\int_0^1 (tq)_\infty \omega(0,\,at)\,d_qt,$$

$$(vi)\quad (q)_\infty \nu(q)=(1-q)^{-1}\int_0^1 (tq)_\infty \nu(0,\,at)\,d_qt,$$

$$(vii)\quad (q)_\infty \rho(q)=(1-q)^{-1}\int_0^1 (tq)_\infty \rho(0,\,at)\,d_qt.$$

4. Certain F_q-Function Associated with the Fifth Order Mock Theta Functions

We now proceed to define certain F_q-functions associated with the ten fifth order mock theta functions as follows, $|q|<1$,

$$(i)\quad f_0(x,\alpha)=\frac{(q)_\infty}{(x)_\infty}\sum_{n=0}^{\infty}\frac{(x)_n}{(q,-q)_n}q^{n^2-n+\alpha n},$$

(4.1)

$$(ii)\quad \Phi_0(x,\alpha)=\frac{(q)_\infty}{(x)_\infty}\sum_{n=0}^{\infty}\frac{(-q;\,q^2)_n(x)_n}{(q)_n}q^{n^2-n+\alpha n},$$

$$(iii) \quad \Psi_0(x,\alpha) = \frac{q(q)_\infty}{(x)_\infty} \sum_{n=0}^{\infty} \frac{(-q,\, x)_n}{(q)_n} q^{\frac{1}{2}n(n+1)+\alpha n},$$

$$(iv) \quad F_0(x,\alpha) = \frac{(q)_\infty}{(x)_\infty} \sum_{n=0}^{\infty} \frac{(x)_n}{(q)_n (q;\, q^2)_n} q^{2n^2-n+\alpha n},$$

$$(v) \quad \chi_0(x,\alpha) = \frac{(q)_\infty}{(x)_\infty} \sum_{n=0}^{\infty} \frac{(x)_n}{(q)_{2n}} (q^\alpha)^n, \quad \mathrm{Re}(\alpha) > 0,$$

$$(vi) \quad f_1(x,\alpha) = \frac{(q)_\infty}{(x)_\infty} \sum_{n=0}^{\infty} \frac{(x)_n}{(q,-q)_n} q^{n^2+\alpha n},$$

$$(vii) \quad \Phi_1(x,\alpha) = \frac{q(q)_\infty}{(x)_\infty} \sum_{n=0}^{\infty} \frac{(-q;\, q^2)_n (x)_n}{(q)_n} q^{n^2+n+\alpha n},$$

$$(viii) \quad \Psi_1(x,\alpha) = \frac{q(q)_\infty}{(x)_\infty} \sum_{n=0}^{\infty} \frac{(-q,\, x)_n}{(q)_n} q^{\frac{1}{2}n(n-1)+\alpha n},$$

$$(ix) \quad F_1(x,\alpha) = \frac{(q)_\infty}{(x)_\infty} \sum_{n=0}^{\infty} \frac{(x)_n}{(q)_n (q;\, q^2)_{n+1}} q^{2n^2+n+\alpha n},$$

$$(v) \quad \chi_1(x,\alpha) = \frac{(q)_\infty}{(x)_\infty} \sum_{n=0}^{\infty} \frac{(x)_n}{(q)_{2n+1}} (q^\alpha)^n, \quad \mathrm{Re}(\alpha) > 0.$$

For $x = q$ and $\alpha = 1$, all the above ten functions (4.1(i-x)) reduces to the ten fifth order mock theta functions $f_0(q)$, $\Phi_0(q)$, $\Psi_0(q)$, $F_0(q)$, $\chi_0(q)$, $f_1(q)$, $\Phi_1(q)$, $\Psi_1(q)$, $F_1(q)$ and $\chi_1(q)$, respectively.

If we take the values of x and α in (4.1(i)-(x)) other than q and 1, respectively, we get the following known standard functions:

$$(i) \quad f_0(0,\alpha) = (q)_\infty (-q^\alpha; q^2)_\infty ,$$

$$(ii) \quad \Phi_0(0,\alpha) = (q)_\infty\, {}_2\Phi_3\begin{bmatrix} i\sqrt{q}, -i\sqrt{q}; q^\alpha \\ 0, \quad 0, \quad 0 \end{bmatrix},$$

$$(iii) \quad \Psi_0(0,\alpha) = q(q^4, q^4)\infty,$$

$$(iv) \quad F_0(q,\alpha-1) =_1 \Phi_2^{(q^2)}\begin{bmatrix} q^2; & q^\alpha \\ q, & 0 \end{bmatrix},$$

$$(v) \quad \chi_0(x,\alpha) = (q)_\infty Cos_q\left(iq^{\alpha/2}\right),$$

$$(vi) \quad f_1(0,\alpha) = (q)_\infty (-q^{\alpha+1}; q^2)_\infty ,$$

$$(vii) \quad \Phi_1(0,\alpha-2) = q(q)_\infty\, {}_2\Phi_3\begin{bmatrix} i\sqrt{q}, -i\sqrt{q}; q^\alpha \\ 0, \quad 0, \quad 0 \end{bmatrix},$$

$$(viii) \quad \Psi_1(0,1) = (q^4, q^4)_\infty ,$$

$$(ix) \quad F_1(q,\alpha-3) = (1-q)^{-1}\, {}_1\Phi_2^{(q^2)}\begin{bmatrix} q^2; & q^\alpha \\ q^3, & 0 \end{bmatrix},$$

$$(v) \quad \chi_1(0,\alpha) = -iq^{-\alpha/2}(q)_\infty Sin_q\left(iq^{\alpha/2}\right).$$

5. Certain q-integral Representations for the F_q-Function Related to the Fifth Order Mock Theta Functions (4.1(i-x))

For $q^\alpha = a$, we may write, for $\operatorname{Re} x > 0, |\alpha| < 1;$

$$(i) \quad f_0\left(q^{\alpha},\alpha\right)=(q)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n^2-n}}{(q,-q)_n \left(q^{x+n}\right)_\infty}$$

(5.1)
$$=(1-q)^{-1}\int_0^1 t^{x-1}(tq)_\infty \sum_{n=0}^{\infty}\frac{(at)^n}{(q,-q)_n} q^{n^2-n} d_q t$$

$$=\frac{(1-q)^{-1}}{(q)_\infty}\int_0^1 t^{x-1}(tq)_\infty f_0(0,\, at)\, d_q t.$$

Similarly, we can write, for $\operatorname{Re} x>0,\ |a|<1;$

$$(ii) \quad \Phi_0\left(q^x,a\right)=\frac{(1-q)^{-1}}{(q)_\infty}\int_0^1 t^{x-1}(tq)_\infty \Phi_0(0,\, at)\, d_q t,$$

$$(iii) \quad \Psi_0\left(q^x,a\right)=\frac{(1-q)^{-1}}{(q)_\infty}\int_0^1 (tq)_\infty \Psi_0(0,\, at)\, d_q t,$$

$$(iv) \quad F_0\left(q^x,a\right)=\frac{(1-q)^{-1}}{(q)_\infty}\int_0^1 (tq)_\infty F_0(0,\, at)\, d_q t,$$

$$(v) \quad \chi_0\left(q^x,a\right)=(1-q)^{-1}\int_0^1 (tq)_\infty Cos\left(i\sqrt{at}\right) d_q t,$$

$$(vi) \quad f_1\left(q^x,a\right)=\frac{(1-q)^{-1}}{(q)_\infty}\int_0^1 t^{x-1}(tq)_\infty f_1(0,\, at)\, d_q t,$$

$$(vii) \quad \Phi_1\left(q^x,a\right)=\frac{(1-q)^{-1}}{(q)_\infty}\int_0^1 t^{x-1}(tq)_\infty \Phi_1(0,\, at)\, d_q t,$$

$$(viii) \quad \Psi_1\left(q^x,a\right)=\frac{(1-q)^{-1}}{(q)_\infty}\int_0^1 t^{x-1}(tq)_\infty \Psi_1(0,\, at)\, d_q t,$$

$$(ix)\quad F_1\left(q^x,a\right)=\frac{(1-q)^{-1}}{(q)_\infty}\int_0^1 t^{x-1}(tq)_\infty F_1(0,\,at)\,d_qt,$$

$$(x)\quad \chi_1\left(q^x,a\right)=\frac{-i(1-q)^{-1}}{\sqrt{a}}\int_0^1 t^{x-3/2}(tq)_\infty\, Sin\left(i\sqrt{\alpha t}\right)d_qt.$$

For $x=1$ and $a=q,$ we have the following q-integral representations for ten fifth order mock theta functions:

(5.2)

$$(i)\quad f_0(q)=(1-q)^{-1}(q)_\infty^{-1}\int_0^1 (tq)_\infty f_0\left(0,\,q^t\right)d_qt,$$

$$(ii)\quad \Phi_0(q)=(1-q)^{-1}(q)_\infty^{-1}\int_0^1 (tq)_\infty \Phi_0\left(0,\,q^t\right)d_qt,$$

$$(iii)\quad \Psi_0(q)=(1-q)^{-1}(q)_\infty^{-1}\int_0^1 (tq)_\infty \Psi_0\left(0,\,q^t\right)d_qt,$$

$$(iv)\quad F_0(q)=(1-q)^{-1}(q)_\infty^{-1}\int_0^1 (tq)_\infty F_0\left(0,\,q^t\right)d_qt,$$

$$(v)\quad \chi_0(q)=(1-q)^{-1}\int_0^1 (tq)_\infty Cos_q\left(i\sqrt{q^t}\right)d_qt,$$

$$(vi)\quad f_1(q)=(1-q)^{-1}(q)_\infty^{-1}\int_0^1 (tq)_\infty f_1\left(0,\,q^t\right)d_qt,$$

$$(vii)\quad \Phi_1(q)=(1-q)^{-1}(q)_\infty^{-1}\int_0^1 (tq)_\infty \Phi_1\left(0,\,q^t\right)d_qt,$$

$$(viii)\quad \Psi_1(q)=(1-q)^{-1}(q)_\infty^{-1}\int_0^1 (tq)_\infty \Psi_1\left(0,\,q^t\right)d_qt,$$

$$(ix)\quad F_1(q)=(1-q)^{-1}(q)_\infty^{-1}\int_0^1 (tq)_\infty F_1\left(0,\, q^t\right)d_q t,$$

$$(x)\quad \chi_1(q)=-iq^{-1/2}(1-q)^{-1}\int_0^1 t^{-1/2}(tq)_\infty Sin_q\left(i\sqrt{\alpha t}\right)d_q t.$$

It is interesting to note the presence of $Cos_q(ix)$ and $Sin_q(ix)$ in the integral of the q-integral representations for $\chi_0(q)$ and $\chi_1(q)$, respectively.

Besides the above integral relations, we can also establish the following impotant q-differential relations between F_q-functions defined in (4.1(i-x)).

$$(5.3)\qquad \begin{aligned}
&(i) && D_{q,x} f_0(x,\alpha)\Big|_{\alpha=1}^{x=q} = f_1(q),\\
&(ii) && D^2_{q,x}\Phi_0(x,\alpha)\Big|_{\alpha=1}^{x=q} = q^{-1}\Phi_1(q),\\
&(iii) && D_{q,x}\Psi_1(x,\alpha)\Big|_{\alpha=1}^{x=q} = q^{-1}\Psi_0(q),\\
&(iv) && D^2_{q,x}F_1(x,\alpha)\Big|_{\alpha=1}^{x=q} = F_0(q)+qF_1(q),\\
&(v) && qD^2_{q,x}\chi_1(x,\alpha)\Big|_{\alpha=1}^{x=q} = \chi_1(q)-\chi_0(q).
\end{aligned}$$

The q-differential relations (5.3(i-v)) are very significant. They clearly exhibit the close relationship between the two group of fifth order mock theta functions. It was asserted by Ramanujan and accepted so, by others that the members of the first group of five functions are only related amongst themselves and the same is true for the second group of five functions. However, (5.3(i-v)) show the close structural, one-one, relationship between members of the first group of five functions with the five members of the second group.

6. F_q-Functions Related to the Mock Theta Functions of Order Seven

It can be easily verified that the following functions related to the seventh order mock theta functions are F_q-Functions:

$$(i)\quad \Im_0(x,\alpha)=\frac{(q)_\infty}{(x)_\infty}\sum_{n=0}^{\infty}\frac{(x)_n}{(q)_{2n}}q^{n^2-n+\alpha n},$$

(6.1)

$$(ii)\quad \Im_1(x,\alpha)=\frac{q(q)_\infty}{(x)_\infty}\sum_{n=0}^{\infty}\frac{(x)_n}{(q)_{2n+1}}q^{n^2+n+\alpha n},$$

$$(iii)\quad \Im_2(x,\alpha)=\frac{(q)_\infty}{(x)_\infty}\sum_{n=0}^{\infty}\frac{(x)_n}{(q)_{2n+1}}q^{n^2+\alpha n}.$$

All the above functions give the corresponding seventh order mock theta function for $x=q$ and $\alpha=1$. In a mannar similar to used for third and fifth order mock theta functions, we can find the integral representations for the F_q-functions related to seventh order mock theta functions.

For $\operatorname{Re} x>0, |a|<1;$ we have

$$(i)\quad (q)_\infty \Im_0(q^x,a)=(1-q)^{-1}\int_0^1 t^{x-1}(tq)_\infty \Im_0(0,\, at)\,d_qt,$$

(6.2)

$$(ii)\quad (q)_\infty \Im_1(q^x,a)=(1-q)^{-1}\int_0^1 t^{x-1}(tq)_\infty \Im_1(0,\, at)\,d_qt,$$

$$(iii)\quad (q)_\infty \Im_2(q^x,a)=(1-q)^{-1}\int_0^1 t^{x-1}(tq)_\infty \Im_2(0,\, at)\,d_qt.$$

For $x=1$ and $a=q,$ we get the integral representations for the seventh order mock theta functions, namely

$$(i)\quad (q)_\infty \Im_0(q)=(1-q)^{-1}\int_0^1 (tq)_\infty \Im_0(0,\, at)\,d_qt,$$

(6.3)

$$(ii)\quad (q)_\infty \Im_1(q)=(1-q)^{-1}\int_0^1 (tq)_\infty \Im_1(0,\, at)\,d_qt,$$

$$(iii)\quad (q)_\infty \Im_2(q)=(1-q)^{-1}\int_0^1 (tq)_\infty \Im_2(0,\, at)\,d_qt,$$

We also have the following two q-differential relations connecting the functions defined in (6.1(i-iii)):

$$(i) \qquad D_{q,x}\Im_2(x,\alpha)\Big|_{\alpha=1}^{x=q} = q^{-1}\Im_1(q),$$

$$(6.4) \qquad (ii) \qquad D_{q,x}\left[\Im_0(x,\alpha)-\Im_1(x,\alpha)\right]\Big|_{\alpha=1}^{x=q} = \Im_2(q).$$

The q-differential relations (6.4(i-ii)) are again very significant. They clearly exhibit the close relationship between the three members of the generalized seventh order mock theta functions. It was asserted by Ramanujan and accepted so, by us that the three seventh order mock theta functions are unrelated. However, (6.4(i-ii)) show the close relationship between the three seventh order mock theta functions.

7. Concluding Remarks

It may be remarked that the generalized F_q-functions constructed by us are not unique. One could construct generalised F_q-functions different from the ones which we have used and which reduce to mock theta functions for particular values of x and α. This study is continued in the next two papers for the mock theta functions found in the 'Lost' Notebook also and for certain expansion formulae for all these generalised functions have been investigated.

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