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Some Properties of a Finsler Space with the Metric $L(x,y)=\gamma(x,y)\phi\left(\frac{\beta}{\gamma}\right)$

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Abstract: In the paper, we consider an n-dimensional Finsler space $F^n\,(n>2)$ with the metric $L(x,y)=\gamma(x,y)\,\phi(s)$, where $s=\frac{\beta}{\gamma}$ and $\phi(s)$ is a differentiable function with respect to s, β is a differential one form and γ is cubic metric. We obtain expressions for the fundamental metric tensor, Cartan tensor, geodesic spray coefficients and the equation of geodesics in a Finsler space with the above metric. Some other properties of such space have also been discussed.

Keywords: Finsler space, cubic metric, geodesics, one form metric.

Mathematics subject classification: 53B40

1. Introduction

The notion of an mth-root metric was introduced by H. Shimada[1] in 1979. By introducing the regularity of the metric, various fundamental quantities of a Finsler metric could be found. In particular, the Cartan connection of a Finsler space with mth-root metric was introduced from the theoretical standpoint. M. Matsumoto and K. Okubo[2] studied Berwald connection of a Finsler space with mth-root metric and gave main scalars in two dimensional case and defined higher order Christoffel symbols. The mth-root metric is used in many problems of theoretical physics[3]. T. N.

Pandey et. al[4] studied three dimensional Finsler space with mth-root metric. To discuss general relativity with the electromagnetic field, G. Randers[5] introduced a metric of the form $L(x,y)=\alpha(x,y)+\beta(x,y)$, where α is a square root metric and β is a differential one form. In his honor, this metric is called Randers metric, and it has been extensively studied by several geometers and physicists[6-8]. Recently P. N. Pandey and Shivalika Saxena[9] studied a Finsler space with the metric L of the form $L(x,y)=F(x,y)+\beta(x,y)$, where F is an mth-root metric and called the space with this metric as an R-Randers mth-root space and obtained many results related to it. In 2010, Ryozo Yoshikawa and Katsumi Okubo[10] studied $S3$-like Finsler spaces with the metric $L(x,y)=\alpha(x,y)\,\phi\left(\dfrac{\beta}{\alpha}\right)$.

The aim of the present paper is to study a more general space with the metric $L(x,y)=\gamma(x,y)\,\phi(s)$, where $s=\dfrac{\beta}{\gamma}$ and $\phi(s)$ is a differentiable function with respect to s, β is a differential one form and γ is cubic metric. The paper is organized as follows. Section 2 deals with some preliminary concepts required for the discussion of the following sections. In section 3, we derive certain identities satisfied in a Finsler space with the above metric. We obtain the fundamental metric tensor g_{ij}, its inverse g^{ij} and the Cartan tensor C_{ijk} for a Finsler space with the metric $L(x,y)=\gamma(x,y)\,\phi(s)$. In section 4, we obtain the spray coefficients of a Finsler space with this metric. It also includes the equations of the geodesics in a Finsler space equipped with the above metric.

2. Preliminaries

Let $F^n=(M,L(x,y))\ (n>2)$ be an n-dimensional Finsler space. The mth-root metric on M is defined as $L^m=a_{i_1 i_2 i_3 \ldots\ldots i_m}(x)y^{i_1}y^{i_2}\ldots\ldots y^{i_m}$, where $a_{i_1 i_2 i_3 \ldots\ldots i_m}(x)$ are components of an mth order covariant symmetric tensor.
In case of $m=2$, the metric L is Riemannian. For $m=3$ and $m=4$ these metrics are called cubic and quartic respectively.
The covariant symmetric metric tensor g_{ij} of $F^n=(M,L(x,y))$ is defined by

$$\text{(2.1)} \qquad g_{ij} =: \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \qquad\qquad \dot{\partial}_i \equiv \frac{\partial}{\partial y^i}.$$

This tensor is positively homogeneous of degree zero in y^i. From the metric tensor g_{ij}, we construct the Cartan tensor C_{ijk} by

(2.2) $$C_{ijk} =: \frac{1}{2}\dot{\partial}_k g_{ij}.$$

The tensor C_{ijk} is symmetric in its lower indices and is positively homogeneous of degree -1 in y^i. Due to its homogeneous and symmetric properties, it satisfies the following

(2.3) $$C_{ijk}y^i = C_{kij}y^i = C_{jki}y^i = 0.$$

The angular metric tensor of a Finsler space is given by

(2.4) $$h_{ij} = g_{ij} - l_i l_j,$$

where $l_i = \dot{\partial}_i L.$

The geodesic of a Finsler space is given by[1]

(2.5) $$\frac{d^2x^i}{dt^2} + 2G^i = 0,$$

where G^i are the geodesic spray coefficients given by

(2.6) $$2G^i = \frac{1}{2}g^{ij}\left\{y^k\dot{\partial}_j\partial_k L^2 - \partial_j L^2\right\}, \qquad \partial_j \equiv \frac{\partial}{\partial x^j}.$$

The nonlinear connection of a Finsler space is defined as

(2.7) $$N^i_j = \dot{\partial}_j G^i.$$

In the present paper, we study the space whose fundamental function is given by

(2.8) $$L(x,y) = \gamma(x,y)\,\phi(s),$$

where

(2.9) $\gamma = \sqrt[3]{a_{lmn}(x)y^l y^m y^n}$ is a cubic metric

(2.10) $\beta = b_i(x)y^i,$

is a differential one form and $s = \dfrac{\beta}{\gamma}$.

3. Fundamental Metric Tensor and Cartan Tensor

In this section, we find the fundamental metric tensor g_{ij}, its inverse g^{ij} angular metric tensor h_{ij} and the Cartan tensor C_{ijk} for a Finsler space with the metric (2.8).

Differentiating (2.9) partially with respect to y^i, we get

(3.1) $$\dot{\partial}_i \gamma = \gamma^{-2} a_i,$$

where $a_i = a_{ijk} y^j y^k$.

Differentiating (2.10) partially with respect to y^i, we find

(3.2) $$\dot{\partial}_i \beta = b_i.$$

Differentiating (2.8) partially with respect to y^i and using (3.1) and (3.2), we have

(3.3) $$\dot{\partial}_i L = \rho a_i + \rho_0 b_i,$$

where

(3.4) $$\rho_0 = \phi' = L_\beta,$$

(3.5) $$\rho = \gamma^{-2}\left(\phi - s\phi'\right)$$

and the subscripts denote the degree of homogeneity of the corresponding entities with respect to y^i.

Differentiating (3.4) and (3.5) partially with respect to y^i and using (3.1) and (3.2), we respectively get

(3.6) $$\dot{\partial}_i \rho_0 = \rho_{-1} b_i + \rho_{-3} a_i,$$

and

(3.7) $$\dot{\partial}_i \rho = \rho_{-3} b_i + \rho_{-5} a_i,$$

where

(3.8) $$\rho_{-1} = L_{\beta\beta} = \gamma^{-1}\phi'',$$

(3.9) $$\rho_{-3} = \gamma^{-1} L_{\gamma\beta} = -s\gamma^{-3}\phi''$$

and

(3.10) $$\rho_{-5} = \gamma^{-4}\left(L_{\gamma\gamma} - 2\gamma^{-1} L_\gamma\right) = \gamma^{-5}\left(s^2\phi'' - 2\phi + 2s\phi'\right).$$

Thus, we have

Proposition 3.1. *In a Finsler space with the metric (2.8), the following identities hold*

$$\dot{\partial}_i \rho_0 = \rho_{-1} b_i + \rho_{-3} a_i \quad \textit{and} \quad \dot{\partial}_i \rho = \rho_{-3} b_i + \rho_{-5} a_i .$$

Differentiating (3.3) partially with respect to y^i and using (2.6) and (2.7), we get

(3.11) $$\dot{\partial}_j \dot{\partial}_i L = 2\rho a_{ij} + \rho_{-1} b_i b_j + \rho_{-3}\left(a_i b_j + a_j b_i\right) + \rho_{-5} a_i a_j .$$

From (2.1), we have

(3.12) $$g_{ij} = \left(\dot{\partial}_i L\right)\left(\dot{\partial}_j L\right) + L\left(\dot{\partial}_j \dot{\partial}_i L\right).$$

Using (3.3) and (3.11), (3.12) yields

(3.13) $$g_{ij} = \left(\rho a_i + \rho_0 b_i\right)\left(\rho a_j + \rho_0 b_j\right) + L\Big(2\rho a_{ij} + \rho_{-1} b_i b_j + \rho_{-3}\left(a_i b_j + a_j b_i\right) + \rho_{-5} a_i a_j\Big).$$

Take $d_{ij} = 2\rho a_{ij}$ and $c_i = q_0 b_i + q_{-2} a_i$, where q_0 and q_{-2} satisfy

$$q_0^{\,2} = L\rho_{-1} + \rho_0^{\,2},\ \ q_{-2}^{\,2} = \rho^2 + L\rho_{-5},\ \ q_0 q_{-2} = \rho\rho_0 + L\rho_{-3}.$$

Then (3.13) takes the form

(3.14) $$g_{ij} = L d_{ij} + c_i c_j .$$

Thus, we have

Theorem 3.2. *The fundamental metric tensor g_{ij} of a Finsler space with the metric (2.8), is given by (3.14).*

Theorem 3.3. *In a Finsler space with the metric (2.8), the inverse g^{ij} of the fundamental metric tensor g_{ij}, is given by*

(3.15) $$g^{ij} = \frac{1}{L}\left(d^{ij} - \frac{1}{L + c^2} c^i c^j\right),$$

where $c^i = d^{ij} c_j$ *and* $c^2 = c^i c_i$.

Proof. Let $\left(d^{ij}\right)$ be the inverse of nonsingular matrix $\left(d_{ij}\right)$. Suppose that $\left(g^{ij}\right)$ is given by (3.15).

Now,

$$g_{ij}g^{jk} = \left(Ld_{ij} + c_i c_j\right)\frac{1}{L}\left(d^{jk} - \frac{1}{L+c^2}c^j c^k\right)$$

$$= d_{ij}d^{jk} - \frac{d_{ij}}{L+c^2}c^j c^k + \frac{1}{L}c_i c_j d^{jk} - \frac{1}{L\left(L+c^2\right)}c_i c_j c^j c^k$$

$$= \delta_i^k - \frac{c_i c^k}{L+c^2} + \frac{1}{L}c_i c^k - \frac{1}{L\left(L+c^2\right)}c^2 c_i c^k$$

$$= \delta_i^k .$$

Therefore, g^{ij} given by (3.15) is the inverse of the matrix g_{ij}. This also shows that $\left(g_{ij}\right)$ is non-degenerate. ▫

Using (3.3) and (3.13) in (2.4), we get the angular metric tensor of a Finsler space with the metric (2.8)

$$h_{ij} = L\left\{2\rho a_{ij} + \rho_{-1}b_i b_j + \rho_{-3}\left(a_i b_j + a_j b_i\right) + \rho_{-5}a_i a_j\right\}. \tag{3.16}$$

Thus, we have

Theorem 3.4. *In a Finsler space F^n with the metric (2.8), the angular metric tensor h_{ij} is given by (3.16).*

Differentiating (3.8), (3.9) and (3.10) partially with respect to y^i, we respectively get

$$\dot{\partial}_i \rho_{-1} = \rho_{-2}b_i + \rho_{-4}a_i , \tag{3.17}$$

$$\dot{\partial}_i \rho_{-3} = \rho_{-4}b_i + \rho_{-6}a_i \tag{3.18}$$

and

$$\dot{\partial}_i \rho_{-5} = \rho_{-6}b_i + \rho_{-8}a_i , \tag{3.19}$$

where $\rho_{-2} = \gamma^{-2}\phi''',\quad \rho_{-4} = -\gamma^{-4}\left(\phi'' + s\phi'''\right),\quad \rho_{-6} = \gamma^{-6}\left(s^2\phi''' + 4s\phi''\right)$ and $\rho_{-8} = -\gamma^{-8}\left(s^3\phi''' + 9s^2\phi'' + 10\phi' - 10\phi\right).$

Thus, we have the following

Proposition 3.5. *In a Finsler space* F^n *with the metric (2.8), the following hold good*

$$\dot{\partial}_i\rho_{-1} = \rho_{-2}b_i + \rho_{-4}a_i,\ \dot{\partial}_i\rho_{-3} = \rho_{-4}b_i + \rho_{-6}a_i,\ \dot{\partial}_i\rho_{-5} = \rho_{-6}b_i + \rho_{-8}a_i.$$

Differentiating (3.13) partially with respect to y^k, we get

$$\begin{aligned}\dot{\partial}_k g_{ij} = &\left[2\rho_0\left(\dot{\partial}_k\rho_0\right) + \left(\dot{\partial}_k b\right)\rho_{-1} + L\left(\dot{\partial}_k\rho_{-1}\right)\right]b_ib_j \\ &+\left(\rho\rho_0 + L\rho_{-3}\right)\left(b_j\dot{\partial}_k a_i + b_i\dot{\partial}_k a_j\right) \\ &+\left(a_ib_j + a_jb_i\right)\left\{\rho\left(\dot{\partial}_k\rho_0\right) + \rho_0\left(\dot{\partial}_k\rho\right) + \rho_{-3}\left(\dot{\partial}_k L\right) + L\left(\dot{\partial}_k\rho_{-3}\right)\right\} \\ &+\left\{2\rho\left(\dot{\partial}_k\rho\right) + \rho_{-5}\left(\dot{\partial}_k L\right) + L\left(\dot{\partial}_k\rho_{-5}\right)\right\}a_ia_j \\ &+\left(\rho^2 + L\rho_{-5}\right)\left\{a_j\left(\dot{\partial}_k a_i\right) + a_i\left(\dot{\partial}_k a_j\right)\right\} \\ &+2\left(\dot{\partial}_k L\right)\rho a_{ij} + 2L\left(\dot{\partial}_k\rho\right)a_{ij} + 2L\rho\left(\dot{\partial}_k a_{ij}\right).\end{aligned} \tag{3.20}$$

Partial differentiation of a_i and a_{ij} with respect to y^k yield

$$\dot{\partial}_k a_i = 2a_{ik} \quad \text{and} \quad \dot{\partial}_k a_{ij} = 2a_{ijk}. \tag{3.21}$$

If we use (3.3), (3.6), (3.7), (3.17), (3.18), (3.19) and (3.21) in (3.20), on simplification it follows that

$$\begin{aligned}2C_{ijk} = &\mu_{-1}b_ib_jb_k + \mu_{-7}a_ia_ja_k + \mu_{-3}\underset{(ijk)}{\sigma} b_ib_ja_k \\ &+\mu_{-5}\underset{(ijk)}{\sigma} a_ia_jb_k + \mu_{-2}\underset{(ijk)}{\sigma} a_{ij}b_k + \mu_{-4}\underset{(ijk)}{\sigma} a_{ij}a_k + 2L\rho a_{ijk},\end{aligned} \tag{3.22}$$

where

$$\mu_{-1} = 3\rho_0\rho_{-1} + L\rho_{-2},\ \ \mu_{-2} = 2\left(\rho\rho_0 + L\rho_{-3}\right),$$
$$\mu_{-3} = 2\rho_0\rho_{-3} + \rho\rho_{-1} + L\rho_{-4},\ \mu_{-4} = 2\left(\rho^2 + L\rho_{-5}\right),$$
$$\mu_{-5} = 2\rho\rho_{-3} + \rho_0\rho_{-5} + L\rho_{-6},\ \mu_{-7} = 3\rho\rho_{-5} + L\rho_{-8}$$

and $\underset{(ijk)}{\sigma}$ denotes the interchange of indices *i, j & k* and addition.

Thus, we have

Theorem 3.6. *In a Finsler space with the metric (2.8), the Cartan tensor* C_{ijk} *is given by (3.22).*

4. Spray and Equation of Geodesics

In this section, we discuss about the spray of a Finsler space with the metric (2.8) and obtain its local coefficients. We also obtain the equation of geodesics in such space.

If we differentiate (2.9) partially with respect to x^l, we get

(4.1) $$\partial_l \gamma = \frac{1}{3}\gamma^{-2} A_l,$$

where

(4.2) $$A_l = \left(\partial_l a_{ijk}\right) y^i y^j y^k.$$

Differentiating (2.10) partially with respect to x^l, we get

(4.3) $$\partial_l \beta = B_k,$$

where $B_k = \left(\partial_l b_i\right) y^i$. If we differentiate (2.8) partially with respect to x^k and use (4.1) and (4.3), it follows that

(4.4) $$\partial_k L^2 = \frac{1}{3}\upsilon_{-1} A_k + \upsilon_1 B_k,$$

where

$$\upsilon_{-1} = \gamma^{-1}\left(2\phi^2 - 2s\phi\phi'\right) \text{ and } \upsilon_1 = 2\phi\phi'\gamma.$$

Further, differentiating (4.4) partially with respect to y^j, we have

(4.5) $$\dot{\partial}_j \partial_k L^2 = \upsilon_{-4} A_k a_j + \upsilon_{-2}\left(\frac{1}{3} A_k b_j + B_k a_j\right) + \upsilon_0 B_k b_j + \upsilon_{-1} A_{kj},$$

where

$$\upsilon_{-4} = -\frac{1}{3}\gamma^{-4}\left(2\phi^2 - 2s^2\phi'^2 - 2s^2\phi\phi'\right),\ \upsilon_{-2} = \gamma^{-2}\left(2\phi\phi' - 2s\phi'^2 - 2s\phi\phi''\right),$$
$$\upsilon_0 = 2\left(\phi'^2 + \phi\phi''\right).$$

In view of (4.4) and (4.5), (2.6) gives

(4.6) $$2G^i = \frac{1}{2} g^{ij}\left[\upsilon_{-4} A_0 a_j + \upsilon_{-2}\left(\frac{1}{3} A_0 b_j + B_0 a_j\right) + \upsilon_0 B_0 b_j + \upsilon_{-1} A_{0j} - \frac{1}{3}\upsilon_{-1} A_j - \upsilon_1 B_j\right],$$

where $A_0 = A_k y^k$, $B_0 = B_k y^k$, $A_{0j} = A_{kj} y^k$ and g^{ij} is given by (3.15).

Thus, we have

Theorem 4.1. *In a Finsler space equipped with the metric (2.8), the spray coefficients are given by (4.6).*

In view of (2.5) and Theorem 4.1, we have

Corollary 4.1. *In a Finsler space with the metric (2.8), the equation of geodesics is given by*

$$\frac{d^2x^i}{dt^2} + 2G^i = 0,$$

where the spray coefficients G^i *are given by (4.6).*

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