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On Finsler Space with Special $(\alpha,\beta)-$Metric

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Abstract: In the present paper we have considered a special $(\alpha,\beta)-$metric $L=\mu\left(\dfrac{\alpha^2}{\alpha-\beta}\right)+\nu(\alpha e^{\beta/\alpha})$, where μ and ν are constants, α is a Riemannian metric and β is a one-form, are given by $\alpha=\sqrt{a_{ij}(x)y^i y^j}$ and $\beta=b_i(x)y^i$. We have found the Berwald connection and the condition under which a Finsler space with this metric is a Berwald space. We have evaluated the main scalars of two dimensional Finsler space with the above metric. The equations of geodesic of the Finsler space with this metric have also been found.

Keywords: Finsler space, special $(\alpha,\beta)-$metric.

2010 Mathematics Subject Classification: 53B40, 53C60.

1. Introduction

M. Matsumoto[1] while studying a Finsler space with $(\alpha,\beta)-$metric of Douglas type, introduced a special $(\alpha,\beta)-$metric given by

$$L=\alpha+\frac{\beta^2}{\alpha}. \tag{1.1}$$

While measuring the slope of mountain with respect to time he[2] introduced a metric given by

$$L=\frac{\alpha^2}{\alpha-\beta}. \tag{1.2}$$

This latter metric is called Matsumoto metric.

Shukla, Prasad and Pandey[3] have considered a special $(\alpha,\beta)-$ metric of Douglas type given by

$$L=\alpha e^{\beta/\alpha}. \tag{1.3}$$

This metric is called exponential $(\alpha,\beta)-$metric.

Shukla and Mishra[4] have considered a special $(\alpha,\beta)-$metric which is a linear combination of metrics (1.1) and (1.2) given by

$$L=A_1\Big(\alpha+\frac{\beta^2}{\alpha}\Big)+A_2\Big(\frac{\alpha^2}{\alpha-\beta}\Big). \tag{1.4}$$

In the present paper we have considered a special $(\alpha,\beta)-$metric which is a linear combination of Matsumoto metric and exponential $(\alpha,\beta)-$metric and is given by

$$L=\mu\Big(\frac{\alpha^2}{\alpha-\beta}\Big)+\nu(\alpha e^{\beta/\alpha}). \tag{1.5}$$

When $\mu=0$ and $\nu\neq 0$, then this metric is homothetic to exponential $(\alpha,\beta)-$metric and when $\mu\neq 0$ and $\nu=0$ then this metric is homothetic to Matsumoto metric. We shall abbreviate by F^n the $n-$dimensional Finsler space with metric (1.5) and R^n the associated Riemannian space.

In the following discussion the Riemannian metric α is not supposed to be positive definite and we shall restrict our discussion to a domain (x, y), where β does not vanish. The covariant differentiation with respect to the Levi-Civita connection $\{\gamma^i_{jk}(x)\}$ of R^n is denoted by the semi-colon (;). We list the symbols here for the latter use[5]:

$$\begin{aligned} &b^i=a^{ir}b_r, && b^2=a^{rs}\,b_r\,b_s, && 2r_{ij}=b_{i;j}+b_{j;i}, && 2s_{ij}=b_{i;j}-b_{j;i},\\ &r^i_j=a^{is}r_{sj}, && s^i_j=a^{ir}s_{rj}, && r_i=b_s r^s_i, && s_i=b_r s^r_i. \end{aligned} \tag{1.6}$$

The Berwald connection $B\Gamma=(G^i_{jk},G^i_j)$ of F^n plays the most important role in the present paper. Let B^i_{jk} denote the difference tensor[6] of G^i_{jk} from γ^i_{jk}

(1.7) $$G^i_{jk}(x,y)=\gamma^i_{jk}(x)+B^i_{jk}(x,y).$$

Contracting (1.7) with respect to y^k and y^j successively, we get

(1.8) $$G^i_j=\gamma^i_{0\,j}+B^i_j, \qquad 2G^i=\gamma^i_{00}+2B^i,$$

where $B^i_j=\dot{\partial}_j\, B^i$, $B^i_{jk}=\dot{\partial}_k\, B^i_j$.

It is to be noted that the Cartan connection and the Berwald connection have the same non-linear connection $\{G^i_j\}$. $B^i(x,y)$ is called difference vector in the present paper and for an $(\alpha,\beta)-$metric it is given by[2]

(1.9) $$B^i=\frac{E}{\alpha}y^i+\frac{\alpha L_\beta}{L_\alpha}s^i_0-\frac{\alpha L_{\alpha\alpha}}{L_\alpha}C^*\Big(\frac{y^i}{\alpha}-\frac{\alpha}{\beta}b^i\Big),$$

where

(1.10) $$E=\frac{\beta L_\beta}{L}C^*, \qquad C^*=\frac{\alpha\beta(r_{00}L_\alpha-2\alpha s_0 L_\beta)}{2(\beta^2L_\alpha+\alpha\gamma^2L_{\alpha\alpha})}, \qquad \gamma^2=b^2\alpha^2-\beta^2.$$

2. Finsler Space with Metric Given by (1.5)

From the metric given by the equation (1.5), we obtain

(2.1) $$L=\frac{H_2}{(\alpha-\beta)}, \qquad L_\alpha=\frac{A_3}{\alpha(\alpha-\beta)^2}, \qquad L_\beta=\frac{B_2}{(\alpha-\beta)^2},$$
$$L_{\alpha\alpha}=\frac{\beta^2C_3}{\alpha^3(\alpha-\beta)^3}, \qquad L_{\alpha\beta}=\frac{-\beta C_3}{\alpha^2(\alpha-\beta)^3}, \qquad L_{\beta\beta}=\frac{C_3}{\alpha(\alpha-\beta)^3},$$

where

(2.2) $$\begin{aligned} H_2&=\mu\alpha^2+\nu\alpha(\alpha-\beta)e^{\beta/\alpha},\\ A_3&=\mu\alpha^2(\alpha-2\beta)+\nu(\alpha-\beta)^3e^{\beta/\alpha},\\ B_2&=\mu\alpha^2+\nu(\alpha-\beta)^2e^{\beta/\alpha},\\ C_3&=2\mu\alpha^3+\nu(\alpha-\beta)^3e^{\beta/\alpha}.\end{aligned}$$

Since $B\Gamma$ is $L-$metrical, i.e. $L_{|i}=L_\alpha\alpha_{|i}+L_\beta\beta_{|i}=0$, therefore from equations (2.1), we have $A_3\alpha_{|i}+\alpha B_2\beta_{|i}=0$, and so

(2.3) $$\alpha_{|i} = -\frac{\alpha B_2}{A_3}\beta_{|i}.$$

It is observed that $\beta_{|i} = b_{s|i}\, y^s = (b_{s;i} - b_r B^r_{si})\, y^s$, which implies that

(2.4) $$\beta_{|i}\, y^i = r_{00} - 2b_r\, B^r.$$

Since $(b^2)_{;i}\, y^i = (\partial_i b^2)\, y^i = 2b^m (r_{mi} + s_{mi})\, y^i$, which gives

(2.5) $$(b^2)_{;i}\, y^i = 2(r_0 + s_0).$$

Now the quadratic form $\gamma^2 = b^2\alpha^2 - \beta^2 = (b^2 a_{ij} - b_i b_j)\, y^i\, y^j$ gives the following result

(2.6) $$A_3(\gamma^2)_{|i}\, y^i = 2A_3(r_0 + s_0)\alpha^2 - 2(B_2 b^2\alpha^2 + A_3\beta)(r_{00} - 2b_r B^r).$$

The following lemma has been used:

Lemma[7] 2.1. *If* $\alpha \equiv 0$ *(mod* β*), that is,* $a_{ij}\,(x)\, y^i\, y^j$ *contains* $b_i(x)\, y^i$ *as a factor, then the dimension is equal to two and* b^2 *vanishes. In this case we have* $\delta = d_i(x)\, y^i$ *satisfying* $\alpha^2 = \beta\delta$ *and* $d_i b^i = 2$.

In the following we consider that $\alpha \not\equiv 0$ (mod β).

3. The Berwald space with Metric (1.5)

Since $L_{|i} = \partial_i L - G^k_i(\dot{\partial}_k L) = 0$, which is written in the form

(3.1) $$A_3 B^k_{ji} y^j y_k = \alpha^2 B_2\,(b_{j;i} - B^k_{ji} b_k)\, y^j,$$

where $y_k = a_{ki} y^i$. Now putting the values of A_3 and B_2 from equation (2.2) in the above equation (3.1) and then rearranging the terms. This equation becomes

(3.2) $$P_3 B^k_{ji} y^j y_k + Q_4\,(b_{j;i} - B^k_{ji} b_k)\, y^j - \alpha[R_2 B^k_{ji} y^j y_k + S_3\,(b_{j;i} - B^k_{ji} b_k)\, y^j] = 0,$$

where P_3, Q_4, R_2 and S_3 are homogeneous functions of degree three, four, two and three respectively in the variables α and β and these are given as follows:

$$\begin{aligned} P_3 &= 2\mu\alpha^2\beta+\nu(3\alpha^2\beta+\beta^3)e^{\beta/\alpha}, \\ Q_4 &= \mu\alpha^4+\nu(\alpha^4+\alpha^2\beta^2)e^{\beta/\alpha}, \\ R_2 &= \mu\alpha^2+\nu(\alpha^2+3\beta^2)e^{\beta/\alpha}, \\ S_3 &= 2\nu\alpha^2\beta\, e^{\beta/\alpha}. \end{aligned} \tag{3.3}$$

Assume that the Finsler space with the metric (1.5) is a Berwald space (art[5]. 25). Then $B^i_{jk} = B^i_{jk}(x)$ and $B^k_{ji}\, y^j\, y_k (= B^k_{ji}\, a_{kh}\, y^j\, y^h)$ is a quadratic form in y^i. Since P_3, Q_4, R_2, S_3 and $b_{j;i}\, y^j$ are rational functions of y^i whereas α is an irrational function of y^i, therefore from equation (3.2), we have

$$P_3B^k_{ji}y^jy_k + Q_4(b_{j;i} - B^k_{ji}b_k)\,y^j = 0, \tag{3.4}$$

and

$$R_2B^k_{ji}y^jy_k + S_3(b_{j;i} - B^k_{ji}b_k)\,y^j = 0. \tag{3.5}$$

If $\alpha^2 \not\equiv 0 \pmod{\beta}$, then $P_3S_3 - Q_4R_2 \neq 0$ and hence

$$B^k_{ji}a_{kh} + B^k_{hi}a_{kj} = 0, \qquad b_{j;i} - B^k_{ji}\,b_k = 0.$$

The former yields $B^k_{ji} = 0$. Consequently the latter gives $b_{j;i} = 0$.

Conversely if $b_{j;i} = 0$, then equation (1.6) gives $r_{ij} = s_{ij} = 0$. Then (1.9) and (1.10) lead to $B^i = 0$, $B^k_{ji} = 0$ and so $G^i_{jk} = \gamma^i_{jk}(x)$. Hence F^n is a Berwald space. Thus we have the following theorem:

Theorem 3.1. *The Finsler space with the metric (1.5) is a Berwald space if and only if the vector b_i is covariantly constant, i.e. $b_{i;j} = 0$ and the Berwald connection is given by $B\Gamma = (\gamma^i_{jk}, \gamma^i_{0j}, 0)$.*

4. Main Scalar of Two Dimensional Finsler Space with $(\alpha,\beta)-$metric (1.5)

The main scalar I of two dimensional Finsler space F^2 with metric $L(\alpha,\beta)$ is given by[7]

$$\epsilon I^2 = \left(\frac{L}{\alpha}\right)^4 \left\{\frac{\gamma^2(T^2)_\beta}{4T^3}\right\}, \tag{4.1}$$

where ϵ is signature of the space, $\gamma^2 = b^2\alpha^2 - \beta^2$,

$$T = p(p + p_0 b^2 + p_{-1}\beta) + \{p_0 p_{-2} - (p_{-1})^2\}\gamma^2, \tag{4.2}$$

$$p = LL_\alpha \alpha^{-12}, \quad p_0 = LL_{\beta\beta} + (L^2)_\beta, \tag{4.3}$$
$$p_{-1} = (LL_{\alpha\beta} + L_\alpha L_\beta)\alpha^{-1}, \quad p_{-2} = L\alpha^{-2}(L_{\alpha\alpha} - L_\alpha \alpha^{-1}) + (L^2)_\alpha \alpha^{-2}.$$

and $T_\beta = \dfrac{\partial T}{\partial \beta}$. If $g = |g_{ij}|$ and $a = |a_{ij}|$ then in $n-$dimensional Finsler space with $(\alpha, \beta)-$metric, we have[8]

$$g = (p^{n-2}T)\,a. \tag{4.4}$$

Substituting the values of L, L_α, L_β, $L_{\alpha\alpha}$, $L_{\alpha\beta}$ and $L_{\beta\beta}$ from (2.1) in (4.3), we get

$$p = \frac{H_2 A_3}{\alpha^2(\alpha-\beta)^3}, \qquad p_0 = \frac{H_2 C_3}{\alpha(\alpha-\beta)^4} + \frac{(B_2)^2}{(\alpha-\beta)^4}, \tag{4.5}$$
$$p_{-1} = \frac{A_3 B_2 \alpha - H_2 C_3 \beta}{\alpha^3(\alpha-\beta)^4}, \qquad p_{-2} = \frac{H_2 C_3 \beta^2 + (A_3)^2\alpha}{\alpha^5(\alpha-\beta)^4} - \frac{H_2 A_3}{\alpha^4(\alpha-\beta)^3},$$

where H_2, A_3, B_2 and C_3 are given in equation (2.2). For a two dimensional Finsler space with $(\alpha, \beta)-$metric, we have[8]

$$\frac{g}{a} = T = \left(\frac{L}{\alpha}\right)^3 \left(L_\alpha + \frac{L_{\beta\beta}}{\alpha}\gamma^2\right). \tag{4.6}$$

Putting the values of L, L_α and $L_{\beta\beta}$ from (2.1) in the above equation (4.6), we get

$$T = \frac{(H_2)^3[A_3\alpha(\alpha-\beta) + C_3\gamma^2]}{\alpha^5(\alpha-\beta)^6}. \tag{4.7}$$

From (4.6) it follows that

$$T_\beta = \frac{\partial T}{\partial \beta} = \frac{3L^2}{\alpha^3} L_\beta \left(L_\alpha + \frac{L_{\beta\beta}}{\alpha}\gamma^2\right) + \left(\frac{L}{\alpha}\right)^3 \left[L_{\alpha\beta} + \frac{1}{\alpha}\left\{\gamma^2 L_{\beta\beta\beta} - 2\beta L_{\beta\beta}\right\}\right], \tag{4.8}$$

where $L_{\beta\beta\beta}=\dfrac{\partial^3 L}{\partial\beta^3}=\dfrac{\nu(\alpha-\beta)^4 e^{\beta/\alpha}+6\mu\alpha^4}{\alpha^2(\alpha-\beta)^4}$. Putting the values of L, L_β, $L_{\alpha\beta}$ and $L_{\beta\beta\beta}$ in equation (4.8), we get

$$(4.9)\qquad T_\beta=\frac{3(H_2)^2 B_2}{\alpha^5(\alpha-\beta)^7}[A_3\alpha(\alpha-\beta)+C_3\gamma^2]-\frac{(H_2)^3}{\alpha^6(\alpha-\beta)^7}\times [3\beta C_3\alpha(\alpha-\beta)-\gamma^2\{\nu(\alpha-\beta)^4 e^{\beta/\alpha}+6\mu\alpha^4\}].$$

Putting the value of L in (4.1) we get the main scalar of two dimensional space with metric (1.5) as

$$(4.10)\qquad \epsilon I^2=\frac{(H_2)^4}{\alpha^4(\alpha-\beta)^4}\Big\{\frac{\gamma^2(T_\beta)^2}{4T^3}\Big\},$$

where H_2, T and T_β are given by (2.2), (4.7) and (4.9) respectively. Thus, we have the following theorem

Theorem 4.1. *The main scalar of two dimensional Finsler space with $(\alpha,\beta)-$metric (1.5) is given by (4.10).*

5. Equations of Geodesic of a Finsler Space with $(\alpha,\beta)-$metric (1.5)

In terms of the arc length s the equations of geodesic of F^n are written in the well known form[7]

$$(5.1)\qquad \frac{d^2x^i}{ds^2}+2G^i\Big(x,\frac{dx}{ds}\Big)=0,$$

where functions $G^i(x,y)$ are given by

$$2G^i=g^{ir}(y^j\dot{\partial}_r\partial_j F-\partial_i F),\quad F=\frac{L^2}{2}.$$

Using the parameter τ, (5.1) may be written as

$$(5.2)\qquad \frac{d^2x^i}{d\tau^2}+2G^i\Big(x,\frac{dx}{ds}\Big)=-\frac{\tau''}{\tau'}\frac{dx^i}{d\tau},$$

where $\tau'=\dfrac{d\tau}{ds}$.

But for an $(\alpha,\beta)-$metric L, the equations of geodesic are given by[8]

(5.3) $$\frac{d^2x^i}{d\tau^2}+\gamma^i_{00}+\frac{2L_\beta}{L_\alpha}s^i_0+\frac{2LL_{\alpha\alpha}E}{L_\alpha L_\beta \beta^2}p^i=0,$$

where $p^i=b^i-\dfrac{\beta}{\alpha^2}y^i$ and E is given by the equation (1.10). Putting the values of L_α, L_β, $L_{\alpha\alpha}$ from (2.1) in the expression of C^* and E given by (1.10), we get

(5.4) $$C^*=\frac{\alpha^2(\alpha-\beta)[r_{00}A_3-2s_0\alpha^2B_2]}{2\beta[\alpha(\alpha-\beta)A_3+\gamma^2C_3]}$$

and

(5.5) $$E=\frac{\alpha^2B_2[r_{00}A_3-2s_0\alpha^2B_2]}{2H_2[\alpha(\alpha-\beta)A_3+\gamma^2C_3]}.$$

Putting the values of L, L_α, L_β, $L_{\alpha\alpha}$ and E in the equation (5.3), we get the equations of geodesic for a Finsler space with metric (1.5) as

(5.6) $$\frac{d^2x^i}{d\tau^2}+\gamma^i_{00}+\frac{2\alpha B_2}{A_3}s^i_0+\frac{C_3[r_{00}A_3-2s_0\alpha^2B_2]}{A_3[\alpha(\alpha-\beta)A_3+\gamma^2C_3]}p^i=0.$$

Taking $\beta=0$, we get the equations of geodesic of the associated Riemannian space R^n as

$$\frac{d^2x^i}{d\tau^2}+\gamma^i_{00}+2s^i_0+\frac{(2\mu\nu)(r_{00}-2s_0\alpha)}{\alpha^2[(\mu+\nu)+b^2(2\mu+\nu)]}p^i=0.$$

References

1. M. Matsumoto, Finsler spaces with $(\alpha,\beta)-$metric of Douglas type, *Tensor, N. S.*, **60** (1998) 123-134.
2. M. Matsumoto, A slope of mountain is a Finsler surface with respect to time measure, *J. Math. Kyoto Univ.*, **29** (1989) 17-25.
3. H. S. Shukla, B. N. Prasad and O. P. Pandey, Exponential change of Finsler metric, *Int. Journal of Contemp. Math. Sciences*, **7 (45-48)** (2012) 2253-2263.
4. H. S. Shukla and S. K. Mishra, On Finsler space with special $(\alpha,\beta)-$metric, *Journal of Int. Acad. of Physical Sciences*, **13(4)** (2009) 337-344.
5. M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Saikawa, Otsu, 520 Japan, 1973.

6. M. Matsumoto, The Berwald connection of Finsler space with an $(\alpha,\beta)-$metric, *Tensor, N. S.,* **50** (1951) 18-21.

7. M. Hashiguchi, S. Hojo and M. Matsumoto, Landsberg spaces of dimension two with $(\alpha,\beta)-$metric, *Tensor N. S.,* **57** (1996) 145-153.

8. M. Kitayama, M. Azuma and M. Matsumoto, On Finsler spaces with $(\alpha,\beta)-$metric, Regularity, geodesics and main scalars, *J. Hokkaido Univ. of Education (Sect.II A),* **46** (1995) 1-10.