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Some Fixed Point Theorems of Semicompatible Mappings

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Abstract: The concept of semicompatible mappings is introduced by Cho, Sharma and Sahu (Semicompatibility and fixed points, Math. Japon, 42 (1995) no. 1, 91-98). In this note we use the concept of semicompatible in metric space to prove some common fixed point theorems.

Keywords: Fixed point, complete metric space, compatible mappings, semicompatible mappings, weakly compatible mappings.

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1. Introduction

The first important result in the theory of fixed point of compatible mappings was obtained by Gerald Jungck[1] in 1986 as a generalization of commuting mappings. Pathak, Chang and Cho[2] in 1994 introduced the concept of compatible mappings of type(P). In 2004 Rohen, Singh and Shambhu[3] introduced the concept of compatible mappings of type(R) by combining the definitions of compatible mappings and compatible mappings of type(P).

The aim of this paper is to prove some common fixed point theorems of semicompatible mappings in metric spaces by considering four self mappings. Following are definition of types of compatible mappings.

Definition[1] 1.1: Let S and T be mappings from a complete metric space X into itself. The mappings S and T are said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition[1] 1.2: Let S and T be mappings from a complete metric space X into itself. The mappings S and T are said to be semicompatible if

$\lim_{n\to\infty} d(STx_n, Tx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition[1] 1.3: Let S and T be mappings from a complete metric space X into itself. If w=Su=Tu for some u in X then w is called a coincidence point of S and T.

Definition[1] 1.4: Let S and T be mappings from a complete metric space X into itself. The mappings S and T are said to be weakly compatible if they commute at their coincidence points, i.e. if $Su = Tu$ for some u in X then $Stu = TSu$.

2. Main Results

Now we give our main theorem.

Theorem 2.4: *Let A, B, S and T be self maps of a complete metric space (X, d) satisfying the following conditions:*

(1) $A(X) \subseteq T(X)$ *and* $B(X) \subseteq S(X)$,

(2) $[d(Ax, Bx)]^2 \leq k_1 d(Ax, Sx)d(By, Ty) + k_2 d(By, Sx)d(Ax, Ty) + k_3 d(Ax, Sx)d(Ax, Ty) + k_4 d(By, Ty)d(By, Sx)$,

where $0 \leq k_1 + k_2 + k_3 + k_4 < 1$; $k_1, k_2, k_3, k_4 \geq 0$.

(3)One of A or S is continuous.

(4)[A, S] is semicompatible and [B, T] is weakly compatible on X.
Then A, B, S and T have a unique common fixed point in X.

Proof: (3) Let $x_0 \in X$ then by (1) there exists $x_1 \in X$ such that $Tx_1 = Ax_0$ and for x_1 there exists $x_2 \in X$ such that $Sx_2 = Bx_1$ and so on. Continuing this process we can define a sequence $\{y_n\}$ in X such that

$$y_{2n+1} = Tx_{2n+1} = Ax_{2n} \text{ and } y_{2n} = Sx_{2n} = Bx_{2n-1}$$

then we have to prove that the sequence $\{y_n\}$ is Cauchy sequence in X.
By condition (2), we have

$$\begin{aligned}[d(y_{2n+1}, y_{2n})]^2 = [d(Ax_{2n}, Sx_{2n-1})]^2 &\leq k_1 d(Ax_{2n}, Sx_{2n})d(Bx_{2n-1}, Tx_{2n-1}) \\ &+ k_2 d(Bx_{2n-1}, Sx_{2n})d(Ax_{2n}, Tx_{2n-1}) + k_3 d(Ax_{2n}, Sx_{2n}) \\ &d(Ax_{2n}, Tx_{2n-1}) + k_4 d(Bx_{2n-1}, Tx_{2n-1})d(Bx_{2n-1}, Sx_{2n}) \\ &= k_1 d(y_{2n+1}, y_{2n})d(y_{2n}, y_{2n-1}) + 0 + k_3 d(y_{2n+1}, y_{2n}) \\ &d(y_{2n+1}, y_{2n-1}) + 0\ [d(y_{2n+1}, y_{2n})] \leq k_1 d(y_{2n}, y_{2n-1}) \\ &+ k_3 [d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1})]\end{aligned}$$

$$[d(y_{2n+1}, y_{2n})] \leq p d(y_{2n}, y_{2n-1}) \text{ where } p = \frac{k_1+k_3}{1-k_3} < 1.$$

Hence $\{y_n\}$ is Cauchy sequence.

Since X is complete so there exists a point $z \in X$ such that $\lim y_n = z$ as $n \to \infty$. Consequently subsequences Ax_{2n}, Sx_{2n}, Bx_{2n-1} and Tx_{2n+1} converges to z. Let S be continuous. Since A and S are semicompatible on X, then by proposition 1.5. we have $S^2x_{2n} \to Sz$ and $ASx_{2n} \to Sz$ as $n \to \infty$.

Now by condition (2) we have

$$\begin{aligned}[d(ASx_{2n}, Bx_{2n-1})]^2 \leq\ & k_1 d(ASx_{2n}, S^2x_{2n}) d(Bx_{2n-1}, Tx_{2n-1}) \\ & + k_2 d(Bx_{2n-1}, S^2x_{2n}) d(ASx_{2n-1}, Tx_{2n-1}) \\ & + k_3 d(ASx_{2n}, S^2x_{2n}) d(ASx_{2n}, Tx_{2n-1}) \\ & + k_4 d(Bx_{2n-1}, Tx_{2n-1}) d(Bx_{2n-1}, S^2x_{2n}).\end{aligned}$$

As $n \to \infty$, we have

$$[d(Sz, z)]^2 \leq k_2 [d(Sz, z)]^2,$$

which is a contradiction. Hence $Sz = z$.

Now
$$\begin{aligned}[d(Az, Bx_{2n-1})]^2 \leq\ & k_1 d(Az, Sz) d(Bx_{2n-1}, Tx_{2n-1}) \\ & + k_2 d(Bx_{2n-1}, Sz) d(Az, Tx_{2n-1}) \\ & + k_3 d(Az, Sz) d(Az, Tx_{2n-1}) \\ & + k_4 d(Bx_{2n-1}, Tx_{2n-1}) d(Bx_{2n-1}, Sz).\end{aligned}$$

Letting $n \to \infty$, we have $[d(Az, z)]^2 \leq k_3 [d(Az, z)]^2$. Hence $Az = z$.

Now since $Az = z$, by condition (1) $z \in T(X)$. Also T is self map of X so there exists a point $u \in X$ such that $z = Az = Tu$. More over by condition (2), we obtain,

$$\begin{aligned}[d(z, Bu)]^2 = [d(Az, Bu)]^2 \leq\ & k_1 d(Az, Sz) d(Bu, Tu) \\ & + k_2 d(Bu, Sz) d(Az, Tu) \\ & + k_3 d(Az, Sz) d(Az, Tu) + k_4 d(Bu, Tu) d(Bu, Sz).\end{aligned}$$

i.e.
$$[d(z, Bu)]^2 \leq k_4 [d(z, Bu)]^2.$$

Hence $Bu = z$ i.e., $z = Tu = Bu$.

Since (B, T) is weak compatible we have $Tz = Bz$.

Now,

$$\begin{aligned}[d(Az, Bz)]^2 \leq\ & k_1 d(Az, Sz) d(Bz, Tz) + k_2 d(Bz, Sz) d(Az, Tz) \\ & + k_3 d(Az, Sz) d(Az, Tz) + k_4 d(Bz, Tz) d(Bz, Sz),\end{aligned}$$

i.e. $[d(Az, Bz)]^2 \leq k_2[d(Az, Bz)]^2$

Hence, $Az = Bz$.

Therefore z is common fixed point of A, B, S and T.
Again let A be continuous then $AAx_{2n} \to Az$ and $ASx_{2n} \to Az$ as $n \to \infty$. Since A and S are semicompatible on X, then we have $ASx_{2n} \to Sz$ as $n \to \infty$. Hence $Az = Sz$.

Now by condition (2) we have

$$[d(AAx_{2n}, Bx_{2n-1})]^2 \leq k_1 d(AAx_{2n}, SAx_{2n})d(Bx_{2n-1}, Tx_{2n-1})$$
$$+k_2 d(Bx_{2n-1}, SAx_{2n})d(AAx_{2n-1}, Tx_{2n-1})$$
$$+ k_3 d(AAx_{2n}, SAx_{2n})d(AAx_{2n}, Tx_{2n-1})$$
$$+ k_4 d(Bx_{2n-1}, Tx_{2n-1})d(Bx_{2n-1}, SAx_{2n}).$$

As $n\to\infty$, we have

$$[d(Az, z)]^2 \leq k_2[d(Az, z)]^2,$$

which is a contradiction.

Hence $Az = z = Sz$.

Now since $Az = z$, by condition (1) $z \in T(X)$. Also T is self map of X so there exists a point $u \in X$ such that $z = Az = Tu$. More over by condition (2), we obtain

$$[d(z, Bu)]^2 = [d(Az, Bu)]^2 \leq k_1 d(Az, Sz)d(Bu, Tu)$$
$$+ k_2 d(Bu, Sz)d(Az, Tu)$$
$$+ k_3 d(Az, Sz)d(Az, Tu)$$
$$+ k_4 d(Bu, Tu)d(Bu, Sz)$$

i.e. $[d(z, Bu)]^2 \leq k_4[d(z, Bu)]^2$.

Hence $Bu = z$ i.e., $z = Tu = Bu$.

Since (B, T) is weak compatible we have $Tz = Bz$.

Now,

$$[d(Az, Bz)]^2 \leq k_1 d(Az, Sz)d(Bz, Tz)$$
$$+ k_2 d(Bz, Sz)d(Az, Tz) + k_3 d(Az, Sz)d(Az, Tz)$$
$$+k_4 d(Bz, Tz)d(Bz, Sz).$$

i.e. $[d(Az, Bz)]^2 \leq k_2[d(Az, Bz)]^2$.

Hence $Az = Bz$.

Therefore z is common fixed point of A, B, S and T.

Finally, in order to prove the uniqueness of z, suppose w be another common fixed point of A, B, S and T Then we have,

$$\begin{aligned}[d(z, w)]^2 = [d(Az, Bw)]^2 \leq k_1[&d(Az, Sz)d(Bw, Tw) \\ &+ d(Bw, Sz)d(Az, Tw)] \\ &+ k_2[d(Az, Sz)d(Az, Tw) \\ &+ d(Bw, Tw)d(Bw, Sz)],\end{aligned}$$

which gives

$$[d(z, Tw)]^2 \leq k_1[d(z, Tw)]^2. \text{ Hence } z = w.$$

This completes the proof.

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