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Absolute Nevanlinna Summability of a Derived Fourier Series

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Abstract: In the present paper, we have studied about Nevanlinna summability of Fourier series. We have proved the theorems of derived Fourier series by generalizing the theorems of *Bosanquet*[1,2], *Samal*[3] for absolute Nevalinna summability.

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1. Introduction

1. Definitions and Notations: *Given a series* $\sum u_n$, *let* $F(w)=\sum_{n<w} u_n$.

Let $q_\delta = q_\delta(t)$ *be defined for* $0 \le t \le 1$.

The $N(q_\delta)$ *transform* $N(F, q_\delta)$ *of F is defined by*

$$N(F,q_\delta)(w)=\int_0^1 q_\delta(t)\,F(wt)\,dt.$$

The series $\sum u_n$ *is said to be summable by the method* $N(q_\delta)$ *to the sum s if*

$$\lim_{w\to\infty} N(F,q_\delta)(w)=s.$$

It is said to be absolutely summable by the method $N(q_\delta)$ *and we shall write*

$$\sum u_n \in \left|N\left(q_\delta\right)\right| \quad if\ N\left(F, q_\delta\right)(w) \in BV(A, \infty).$$

For some $A \geq 0$, which is indeed equivalent to

$$\int_A^\infty \left| \sum_{n<w} q_\delta\left(\frac{n}{w}\right) n\, u_n \right| \frac{dw}{w^2} < \infty$$

For the regularity, we need

$$\int_0^1 q_\delta(t)\, dt = 1$$

The parameter δ will be a non-negative real number. We have further two sets of restriction on q_δ: one for $0 \leq \delta \leq 1$ and the other for $\delta \geq 1$.

In the case $0 \leq \delta \leq 1, q_\delta(t)$ is increasing for $0 < t < 1$.

In the case $\delta \geq 1, q_\delta$ satisfies following $q_\delta(t)$ is decreasing for $0 < t < 1$ with $p = [\delta]$, the integral part of δ,

$$\left[\frac{d}{dt}\right]^{p-1} q_\delta(t) \in A \subset [0,1]$$

$$\left[\left(\frac{d}{dt}\right)^k q_\delta(t)\right]_{t=1} = 0, \qquad k = 0, 1, 2 \ldots\ldots (p-1)$$

$$(-1)^p \left(\frac{d}{dt}\right)^p q_\delta(t) \geq 0$$

and is increasing

Also for $\delta \geq 0$, p $= [\delta]$, we assume

$$\frac{Q_\delta(t)}{t^{\delta - p + 1}} \in L(0,1).$$

where

$$Q_\delta(t) = \int_{(1-t)}^{1} q_\delta^{(p)}(x)\,dx .$$

Let $f(t)$ be a periodic function with period 2π and Lebesgue intregrable over $(-\pi,\pi)$ and let

$$(2.1) \quad f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

The first differentiated series of (2.1) at t=x is

$$(2.2) \quad \sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x) \text{ we get}$$

$$\psi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}$$

$$g(t) = \frac{\psi(t)}{t}$$

$$g(n,t) = \int_t^{\pi} (y-t)^{n-\alpha} \cos\left(ny - \frac{n\pi}{2}\right) dy,$$

where $h = [\alpha]$, the integral part of α

$$H^*(n,t,\alpha) = \frac{1}{\Gamma(\alpha+1)} \int_t^{\pi} \frac{d}{dv} g(n,v)\,dv$$

and $\quad H(n,t) = H^*(n,t,0)$

3. Generalizing the theorems of *Bosanquet*[1,2], *Samal*[3] has proved the following theorem .

Theorem A: *Let* $1 > c > 0$. *Let the function* q_c *satisfy the Conditions*

$$\int_0^1 q_\delta(t) = 1$$

and $0 \le \delta \le 1$, $q_c(t)$ is increasing for $0 < t < 1$ *and let*

$Q_c(t)/t^{c+1} \in L(0,1)$, *then*

$$\int_0^{\pi} t^{-c} \left| d\Phi(t) \right| < \infty = \sum \left| N(q_c) \right|$$

In 2000 *Dikshit*[4] extended the above result for absolute Nevanlinna summability of Fourier series as follows[4] :

Theorem B: *Let* $\alpha \geq 0$ *and let the functions* q_α *satisfy the conditions*

$$\int_0^1 q_\delta(t)\, dt = 1$$

for $\delta \geq 0, p = [\delta]$, *we assume*

$$\frac{Q_\delta(t)}{t^{\delta-p+1}} \in L(0,1)$$

where

$$Q_\delta(t) = \int_{(1-t)}^{1} q_\delta{}^{(p)}(x)\, dx \quad .$$

with $\delta = \alpha$. *If* $\phi_\alpha(t) \in BV(0,\pi)$, *then at* $t = x$ *the fourier series*

of f is summable by the method $\left| N(q_\alpha) \right|$.

The object of the present paper is to extend the above theorems for absolute Nevalinna summability of derived Fourier series :

Main Theorem: We establish our result in the form of the following theorem :

Theorem : Let $\alpha \geq 0, 1 < p \leq 2, \ \alpha > \frac{1}{p}$ and

let the functions q_α *satisfy the conditions*

$$(4.1) \qquad \int_0^1 q_\delta(t)dt = 1,$$

for $\delta \geq 0$, $p = [\delta]$, *we assume* $\dfrac{Q_\delta(t)}{t^{\delta-p+1}} \in L(0,1)$.

where

$$(4.2) \qquad Q_\delta(t) = \int_{(1-t)}^{1} q_\delta{}^{(p)}(x)\,dx.$$

and $\chi(t)$ *is of bounded variation in* $(0,\pi)$ *such that*

$$(4.3) \qquad \int_0^\pi \frac{Q(t)}{t\chi(t)}dt = 0\left[\frac{t}{\log\frac{1}{t}}\right] \qquad (t \to +\infty)$$

then at $t = x$ *the derived series of a Fourier series of f is summable by the method* $|N(q_n)|$.

Proof: Let $T_n^\mu(x)$ denote the sum of
the first n terms of the series (1.2) at the point $t = x$.
Then we have

$$T_n^\mu(x) = \frac{-1}{2\pi}\int_0^\pi \{f(x+t) - f(x-t)\}\frac{d}{dt}\left\{\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{\sin\frac{1}{2}t}\right\}N(F,q_\delta)(w)\,dt$$

where $N(F,q_\delta)(w)$ is the Nevanlinna mean of the sequence $\{\sin nt\}$.
Now, on integration by parts, we obtain

$$T_n^\mu(x) = \frac{1}{2\pi}\int_0^\pi \left\{\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right\}N(F,q_\delta)(w)\,d\{f(x+t) - f(x-t)\}$$

$$T_n^{\mu}(x)=\frac{1}{2\pi}\int_0^{\pi}\left\{\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right\}N(F,q_\delta)(w)\,dg(t)+f^{'}(x)$$

hence

$$T_n^{\mu}(x)-f^{'}(x)=\frac{1}{2\pi}\int_0^{\pi}\left\{\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right\}N(F,q_\delta)(w)\,dg(t)$$

By using definitions, we obtain

$$T_n-f^{'}(x)=\frac{1}{Q_\delta(t)}\sum_{k=o}^{n}q_\delta(k)\left\{T_{n-k}^{\mu}(x)-f^{'}('x)\right\}\frac{1}{x}\left(1og\frac{1}{(1-x)}\right)$$

$$=\frac{1}{Q_\delta(t)}\sum_{k=0}^{n}q_\delta(k)\frac{d}{dt}\left[\frac{\cos\left(\frac{t}{2}\right)-\cos\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\right]N(F,q_\delta)(w)\left(\frac{1}{t}\right)\left(1\log\frac{1}{(1-t)}\right)$$

$$=\int_0^{\pi}\frac{1}{2\pi Q_\delta(t)}\sum_{k=0}^{n}q_\delta(k)\left[\frac{\cos\left(\frac{t}{2}\right)-\cos\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\right]N(F,q_\delta)(w)\left(\frac{1}{t}\right)\left(1og\frac{1}{(1-t)}\right)dg(t)$$

$$=\left[\int_0^{\frac{1}{n}}+\int_{\frac{1}{n}}^{\delta}\right]dg(t)\frac{1}{2\pi Q_\delta(t)}\sum_{k=0}^{n}q_\delta(k)\frac{\cos\left(n-k+\frac{t}{2}\right)}{\cos\left(\frac{t}{2}\right)}N(F,q_\delta)(w)\left(\frac{1}{t}\right)\left(1og\frac{1}{(1-t)}\right)+o(1)$$

$for\,(0<\delta<\pi)$

$(5.1)\ T_n-f^{'}(x)=I_1+I_2+o(1)$ say

Now

$$I_1 = \frac{1}{2\pi Q_\delta(n)} \int_0^{\frac{1}{n}} dg(t) \sum_{k=0}^{n} q_\delta(k) \frac{\cos\left(n-k+\frac{1}{2}\right)t}{\cos\left(\frac{t}{2}\right)} N(F,q_\delta)(w)\left(\frac{1}{2}\right) 1og\left(\frac{1}{(1-t)}\right)$$

$$= o\left(\frac{1}{Q_\delta(n)}\right) \int_0^{\frac{1}{n}} dg(t) \sum_{k=0}^{n} q_\delta(k) \frac{\cos\left(n-k+\frac{1}{2}\right)t}{\cos\left(\frac{t}{2}\right)} N(F,q_\delta)(w)\left(\frac{1}{t}\right) 1og\left(\frac{1}{(1-t)}\right)$$

$$= o\left(\frac{1}{Q_\delta(n)}\right) \int_0^{\frac{1}{n}} |dg(t)| \left| \sum_{k=o}^{n} q_\delta(k) \frac{\cos\left(n-k+\frac{1}{2}\right)t}{\cos\left(\frac{t}{2}\right)} N(F,q_\delta)(w) \right| \left| \left(\frac{1}{t}\right) 1og\left(\frac{1}{(1-t)}\right) \right|$$

$$= o(n) \int_0^{\frac{1}{n}} |dg(t)| \left|\frac{1}{t}\right| \left| 1og\frac{1}{(1-t)} \right|$$

$$= o\left[\frac{nt}{\chi\left(\frac{1}{t}\right)}\right]_0^{1/n}$$

$$= o\left[\frac{1}{\chi(n)}\right]$$

(5.2) $\qquad I_1 = o(1)$

Now, for $\quad \frac{1}{n} \le t \le \delta, \quad$ we have

$$\sum_{k=0}^{n} q_\delta(k) \frac{\cos\left(n-k+\frac{1}{2}\right)t}{\cos\left(\frac{t}{2}\right)} N(F,q_\delta)(w)\left(\frac{1}{t}\right) 1og\left(\frac{1}{(1-t)}\right)$$

$$= o\left[t^{-1}Q_\delta\left(\frac{1}{t}\right)\right] \text{ by virtue of conditions.}$$

$I_2 =$

$$= o\left(\frac{1}{Q_\delta(n)}\right)\int_{1/n}^{\delta}|dg(t)|\left|\sum_{k=0}^{n} q_\delta(k)\frac{\cos\left(n-k+\frac{1}{2}\right)t}{\cos\left(\frac{t}{2}\right)}N(F,q_\delta)(w)\left(\frac{1}{t}\right)\log\left(\frac{1}{(1-t)}\right)\right|$$

$$= o\left(\frac{1}{Q_\delta(n)}\right)\int_{1/n}^{\delta}|dg(t)|\left|t^{-1}Q_\delta\left(\frac{1}{t}\right)N(F,q_\delta)(w)\left(\frac{1}{t}\right)\log\left(\frac{1}{(1-t)}\right)\right|$$

$$= o\left(\frac{1}{Q_\delta(n)}\right)\int_{1/n}^{\delta}|dg(t)|\frac{\left(\frac{1}{t}\right)Q_\delta\left(\frac{1}{t}\right)\chi(t)}{t\chi(t)}N(F,q_\delta)(w)$$

$$= o\left(\frac{1}{\chi(n)Q_\delta(n)}\right)+o\left(\frac{1}{Q_\delta(n)}\right)\left[H(n,t)t^{-1}Q_\delta\left(\frac{1}{t}\right)N(F,q_\delta)(w)\right]_{\left(\frac{1}{n}\right)}^{\delta}$$

$$+o\left(\frac{1}{Q_\delta(n)}\right)\int_{\frac{1}{n}}^{\delta}o\left(\frac{t}{\chi(1/t)}\right)H(n,t)\,d\left\{\frac{Q_\delta\left(\frac{1}{t}\right)}{t\chi\left(\frac{1}{t}\right)}\right\}N(F,q_\delta)(w)+o(1)$$

$$= o\left(\frac{1}{\chi(n)Q_\delta(n)}\right)+o\left(\frac{1}{\chi(n)}\right)$$

$$+\left(\frac{1}{Q_\delta(n)}\right)\int_{\frac{1}{n}}^{\delta}o\left(\frac{t}{\chi(1/t)}\right)H(n,t)N(F,q_\delta)(w)\frac{Q_\delta\left(\frac{1}{t}\right)}{t\chi\left(\frac{1}{t}\right)}d\chi\left(\frac{1}{t}\right)$$

$$+o\left(\frac{1}{Q_\delta(n)}\right)\int_{\frac{1}{n}}^{\delta}o\left(\frac{t}{\chi(1/t)}\right)H(n,t)N(F,q_\delta)(w)\left\{\frac{Q_\delta\left(\frac{1}{t}\right)}{t\chi\left(\frac{1}{t}\right)}\right\}\chi\left(\frac{1}{t}\right)+o(1)$$

$$= o\left(\frac{1}{\chi(n)}\right) + o\left(\frac{1}{\chi(n)Q_\delta(n)}\right) + o\left(\frac{1}{Q_\delta(n)}\right)\int_{\frac{1}{n}}^{\delta} \frac{H(n,t)N(F,q_\delta)(w)}{\left\{\frac{1}{t}\chi\left(\frac{1}{t}\right)\right\}^2} d\chi\left(\frac{1}{t}\right)$$

$$+ o\left(\frac{1}{Q_\delta(n)}\right)\int_{\frac{1}{n}}^{\delta} tN(F,q_\delta)(w)H(n,t)\,d\left\{\frac{Q_\delta\left(\frac{1}{t}\right)}{t\chi\left(\frac{1}{t}\right)}\right\} + o(1)$$

$$= o\left(\frac{1}{\chi(n)}\right) + \left(\frac{1}{\chi(n)Q_\delta(n)}\right) + o(1)\left(\frac{1}{\chi\left(\frac{1}{t}\right)}\right)_{\left(\frac{1}{n}\right)}^{\delta},$$

$$+ o\left(\frac{1}{Q_\delta(n)}\right)\left\{\left(\frac{H(n,t)Q_\delta\left(\frac{1}{t}\right)}{\frac{1}{t}\chi\left(\frac{1}{t}\right)}\right)_{\left(\frac{1}{n}\right)}^{\delta} - \int_{\frac{1}{n}}^{\delta} \frac{H(n,t)Q_\delta\left(\frac{1}{t}\right)N(F,q_\delta)(w)\,dt}{\left(\frac{1}{t}\right)\chi\left(\frac{1}{t}\right)}\right\} + o(1)$$

$$= o\left(\frac{1}{\chi(n)}\right) + o\left(\frac{1}{\chi(n)Q_\delta(n)}\right) + o\left(\frac{1}{Q_\delta(n)}\right)\int_{\frac{1}{n}}^{\delta} \frac{Q_\delta\left(\frac{1}{t}\right)H(n,t)N(F,q_\delta)(w)}{t\chi\left(\frac{1}{t}\right)} dt + o(1)$$

$$= o\left(\frac{1}{\chi(n)}\right) + o\left(\frac{1}{\chi(n)Q_\delta(n)}\right) + o\left(\frac{1}{Q_\delta(n)}\right) + o\left(\frac{1}{Q_\delta(n)}\right)o\left(Q_\delta(n)\right) + o(1)$$

(5.3) $I_2 = o(1)$ by the virtue of conditions of theorem.

Finally the proof of the theorem is completed by considering (5.1), (5.2) and (5.3).

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