ISSN 0974 - 9373

Vol. 18 No.1 (2014) **Journal of International Academy of Physical Sciences** pp. 47- 57

A Note on Special Projective Semi-Symmetric Connection

S. K. Pal
Department of Mathematical Sciences, A.P.S. University, Rewa, India, 486003
E-mail: skpalmath85@gmail.com

M. K. Pandey
Department of Mathematics, University Institute of Technology
Rajiv Gandhi Proudyogiki Vishwavidyalaya Bhopal, India, 462036
E-mail: mkp_apsu@rediffmail.com

R. N. Singh
Department of Mathematical Sciences, A.P.S. University, Rewa, India, 486003
E-mail: rnsinghmp@rediffmail.com

(Received January 05, 2014)

Abstract: In the present paper, we have studied some properties of curvature tensor and Ricci tensor of special projective semi-symmetric connection. It has been shown that if torsion tensor of M^n is covariant constant, then manifold admits a parallel vector field.

AMS Mathematics Subject Classification (2010): 53C12.

Keywords and phrases: Projective semi-symmetric connection, curvature tensor.

1. Introduction

The idea of semi-symmetric connection was introduced by A. Friedmann and J. A. Schouten[1] in 1924. In 1932, H. A. Hayden [2] studied semi-symmetric metric-connection. It was K. Yano[3] who started systematic study of semi-symmetric metric connection and this was further studied by T. Imai [4], R. S. Mishra and S. N. Pandey [5], U. C. De and B. K. De [6] and several other mathematicians [7,8] , In 2001, P. Zhao and H. Song [9] studied a semi-symmetric connection which is projectively equivalent to Levi-Civita connection. Such a connection is called as projective semi-symmetric connection. They found an invariant under the transformation of projective semi-symmetric connection and showed that this invariant could degenerate into the Weyl projective curvature tensor under certain conditions. After this various papers [10,11,12] on projective semi-symmetric metric connection have appeared.

The organization of the paper is as follows. After introduction we give some preliminary results in section 2. In section 3, we present a brief account of special projective semi-symmetric connection and some results concerning torsion tensor, Ricci tensor and curvature tensor.

2. Preliminaries

Let M^n be an n-dimensional $(n > 2)$ Riemannian manifold equipped with a Riemannian metric g and ∇ be the Levi-Civita connection associated with metric g. A linear connection $\overline{\nabla}$ on M^n is said to be semi-symmetric connection if its torsion tensor $\overline{T}$, given by

(2.1) $$\overline{T}(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y]$$

satisfies the condition

(2.2) $$\overline{T}(X,Y) = \pi(Y)X - \pi(X)Y$$

and

(2.3) $$(\overline{\nabla}_X g)(Y,Z) = 0,$$

where π is a 1 - form on M^n associated with vector field ρ i.e.,

(2.4) $$\pi(X) = g(X,\rho).$$

If the geodesic with respect to $\overline{\nabla}$ are always consistent with those of ∇, then $\overline{\nabla}$ is called a connection projectively equivalent to ∇. If $\overline{\nabla}$ is projective equivalent connection to ∇ as well as semi-symmetric, then $\overline{\nabla}$ is called projective semi- symmetric connection. We also call $\overline{\nabla}$ as projective semi- symmetric transformation.

In this paper, we study a type of projective semi-symmetric connection $\overline{\nabla}$ introduced by P.Zhao and H. Song [9]. The connection is given by

(2.5) $$\overline{\nabla}_X Y = \nabla_X Y + \psi(Y)X + \psi(X)Y + \phi(Y)X - \phi(X)Y,$$

where 1-forms ϕ and ψ are given as

(2.6) $$\phi(X) = \frac{1}{2}\pi(X) \; and \; \psi(X) = \frac{n-1}{2(n+1)}\pi(X).$$

It is easy to observe that torsion tensor of projective semi-symmetric transformation is same as given by the equation (2.2) and also that

(2.7) $$(\overline{\nabla}_X g)(Y,Z) = \frac{1}{n+1}[2\pi(X)g(Y,Z) - n\pi(Y)g(Z,X) - n\pi(Z)g(X,Y)].$$

i.e., the connection $\overline{\nabla}$ is a non metric one.

Let $\overline{R}$ and R be the curvature tensors of the manifold relative to the projective semi-symmetric connection $\overline{\nabla}$ and Levi-Civita connection ∇ respectively. It is known that [9]

(2.8) $$\overline{R}(X,Y,Z) = R(X,Y,Z) + \beta(X,Y)Z + \alpha(X,Z)Y - \alpha(Y,Z)X,$$

where $\beta(X,Y)$ and $\alpha(X,Y)$ are given by the following relations

(2.9) $$\beta(X,Y) = \Psi'(X,Y) - \Psi'(Y,X) + \Phi'(Y,X) - \Phi'(X,Y),$$

(2.10) $$\alpha(X,Y) = \Psi'(X,Y) + \Phi'(Y,X) - \psi(X)\phi(Y) - \phi(X)\psi(Y),$$

(2.11) $$\Psi'(X,Y) = (\nabla_X \psi)(Y) - \psi(X)\psi(Y)$$

and

(2.12) $$\Phi'(X,Y) = (\nabla_X \phi)(Y) - \phi(X)\phi(Y).$$

Contracting X in the equation (2.8), we get a relation between Ricci tensors $\overline{R}ic(Y,Z)$ and $Ric(Y,Z)$ of manifold with respect to the connections $\overline{\nabla}$ and ∇ respectively as

(2.13) $$\overline{R}ic(Y,Z) = Ric(Y,Z) + \beta(Y,Z) - (n-1)\alpha(Y,Z).$$

If $\overline{r}$ and r are scalar curvatures of manifold with respect to connections $\overline{\nabla}$ and ∇ respectively, then from the equation (2.13), we get

(2.14) $$\overline{r} = r + b - (n-1)a,$$

where

$$b = \sum_{i=1}^{n} \beta(e_i,e_i) \text{ and } a = \sum_{i=1}^{n} \alpha(e_i,e_i).$$

3. Main Results

In this section, we consider a type of projective semi-symmetric connection $\overline{\nabla}$ given by the equation (2.5) whose associated 1-form π is closed, i.e.,

$$(\overline{\nabla}_X \pi)Y = (\overline{\nabla}_Y \pi)X. \tag{3.1}$$

In this case $\overline{\nabla}$ is called special projective semi-symmetric connection [9].

It is easy to verify that both the 1-forms ϕ and ψ are closed as the 1-form π is closed and we easily get the tensors Φ' and Ψ' both are symmetric. Consequently, we get

$$\beta(X,Y) = 0 \tag{3.2}$$

and

$$\alpha(X,Y) = \alpha(Y,X). \tag{3.3}$$

In view of the equations (3.1) and (3.2) the expressions (2.8), (2.13) and (2.14) reduces to

$$\overline{R}(X,Y,Z) = R(X,Y,Z) + \alpha(X,Z)Y - \alpha(Y,Z)X, \tag{3.4}$$

$$\overline{R}ic(Y,Z) = Ric(Y,Z) - (n-1)\alpha(Y,Z) \tag{3.5}$$

and

$$\overline{r} = r - (n-1)a. \tag{3.6}$$

It is easy to observe that the Ricci tensor $\overline{R}ic(Y,Z)$ is symmetric.

Consequently, from the equations (2.6) and (2.10) ,we have

$$\alpha(X,Y) = \frac{1}{2}(c+1)[(\nabla_X \pi)(Y) - \frac{1}{2}(c+1)\pi(X)\pi(Y)]. \tag{3.7}$$

Differentiating the torsion tensor of the connection $\overline{\nabla}$ given by the equation (2.2) covariantly with respect to the connection $\overline{\nabla}$, we have

$$(\overline{\nabla}_X \overline{T})(Y,Z) = (\overline{\nabla}_X \pi)(Z)Y - (\overline{\nabla}_X \pi)(Y)Z. \tag{3.8}$$

Now, due to the equations (2.5) and (2.6) , we have

(3.9) $$(\overline{\nabla}_X \pi)Y = (\nabla_X \pi)(Y) - c\pi(X)\pi(Y),$$

where $c = \dfrac{n-1}{n+1}.$

Theorem 3.1 *The torsion tensor of special projective semi-symmetric connection satisfies*

$$(\overline{\nabla}_X \overline{T})(Y, Z) = 0$$

if and only if $\alpha(X, Y) = m\pi(X)\pi(Y),$ *where* $m = \dfrac{c^2 - 1}{4}.$

Proof: First suppose that

(3.10) $$(\overline{\nabla}_X \overline{T})(Y, Z) = 0.$$

Therefore from the equation (3.8), we have

$$(\overline{\nabla}_X \pi)(Z)Y - (\overline{\nabla}_X \pi)(Y)Z = 0,$$

which on contraction, gives

(3.11) $$(\overline{\nabla}_X \pi)(Y) = 0.$$

Now from the equation (3.9), we get

$$(\nabla_X \pi)(Y) = c\pi(X)\pi(Y).$$

Again using this in the equation (3.7), we have

(3.12) $$\alpha(X, Y) = m\pi(X)\pi(Y).$$

Conversely, we suppose that α satisfies the equation (3.12). Now, using equation (3.12) in the equation (3.7), we get

$$(\nabla_X \pi)(Y) = c\pi(X)\pi(Y),$$

which on using in the equation (3.9), gives $(\overline{\nabla}_X \pi)(Y) = 0.$

Thus, due to this the equation (3.8), gives $(\overline{\nabla}_X \overline{T})(Y, Z) = 0.$

This completes the proof.

Theorem 3.2 *The associated* 1*-form of special projective semi-symmetric connection satisfies* $(\overline{\nabla}_X \pi)Y = \frac{1-c}{2}\pi(X)\pi(Y)$ if and only if tensor α vanishes.

Proof: If Ricci tensor of a flat Riemannian manifold with respect to special projective semi-symmetric connection vanishes, then from the equation (3.5), we have

(3.13) $$\alpha(Y,Z)=0.$$

Thus, the equation (3.7) reduces to $(\nabla_X \pi)(Y) = \frac{1}{2}(c+1)\pi(X)\pi(Y)$.

Using this in the equation (3.9), we have

(3.14) $$(\overline{\nabla}_X \pi)Y = \frac{1-c}{2}\pi(X)\pi(Y).$$

Conversely, suppose that (3.14) holds. Then from equation (3.9), we have

$$(\nabla_X \pi)(Y) = \frac{1}{2}(c+1)\pi(X)\pi(Y)].$$

Using this in the equation (3.7) ,we get $\alpha(X,Y)=0$.

This competes the proof.

Theorem 3.3 *The Ricci tensor of special projective semi-symmetric connection vanishes if and only if* $W(X,Y,Z)=\overline{R}(X,Y,Z)$.

Proof: The Weyl curvature tensor of Riemannian manifold[13] is given by

(3.15) $$W(X,Y,Z)=R(X,Y,Z)-\frac{1}{n-1}\{Ric(Y,Z)X-Ric(X,Z)Y\}.$$

Suppose that

(3.16) $$W(X,Y,Z)=\overline{R}(X,Y,Z).$$

Then from the equation (3.15), we have

$$\overline{R}(X,Y,Z)=R(X,Y,Z)-\frac{1}{n-1}\{Ric(Y,Z)X-Ric(X,Z)Y\}.$$

In view of equation (3.4), above equation takes the form

$$[Ric(Y,Z)-(n-1)\alpha(Y,Z)]X=[Ric(X,Z)-(n-1)\alpha(X,Z)]Y.$$

Using the equation (3.5) in above equation, we have

$$\overline{R}ic(Y,Z)X=\overline{R}ic(X,Z)Y,$$

which on contraction , gives $\overline{R}ic(Y,Z)=0.$

Conversely, suppose that

(3.17 $\qquad \overline{R}ic(Y,Z)=0.$

Then from the equation (3.5) , we have

$$Ric(Y,Z)=(n-1)\alpha(Y,Z).$$

Using this in the equation (3.15) , we have

$$W(X,Y,Z)=R(X,Y,Z)+\alpha(X,Z)Y-\alpha(Y,Z)X,$$

which in view of the equation (3.4) gives

$$W(X,Y,Z)=\overline{R}(X,Y,Z).$$

This completes the proof.

Theorem 3.4 *The torsion tensor of special projective semi-symmetric connection in manifold M^n is recurrent if and only if 1-form π is recurrent with respect to special projective semi-symmetric connection.*

Proof: Suppose that

(3.18) $\qquad (\overline{\nabla}_X\overline{T})(Y,Z)=\pi(X)\overline{T}(Y,Z).$

Using the equations (2.2) and (3.8) in above equation, we have

$$(\overline{\nabla}_X\pi)(Z)Y-(\overline{\nabla}_X\pi)(Y)Z=\pi(X)[\pi(Z)Y-\pi(Y)Z].$$

In view of equation (3.9) , above equation takes the form

$$(\nabla_X\pi)(Z)Y-(\nabla_X\pi)(Y)Z=(c+1)\pi(X)[\pi(Z)Y-\pi(Y)Z],$$

which on contraction, gives

(3.19) $\qquad (\nabla_X\pi)(Z)=(c+1)\pi(X)\pi(Z).$

The above equation can be written as

$$(\nabla_X\pi)(Z)-c\pi(X)\pi(Z)=\pi(X)\pi(Z).$$

Using the equation (3.9) in above, we have

(3.20) $\qquad (\overline{\nabla}_X\pi)(Z)=\pi(X)\pi(Z),$

which shows that 1-form π is recurrent with respect to special projective semi-symmetric connection.

Conversely, suppose that the equation (3.20) holds. Then we have

$$(\overline{\nabla}_X \pi)(Z)Y - (\overline{\nabla}_X \pi)(Y)Z = \pi(X)\pi(Z)Y - \pi(X)\pi(Y)Z \, .$$

Now, using the equations (3.8) and (2.2) in the above equation, we get

$$(\overline{\nabla}_X T)(Z,Y) = \pi(X)\overline{T}(Y,Z).$$

This completes the proof.

Theorem 3.5 *If the torsion tensor of special projective semi-symmetric connection is recurrent with π as 1-form of recurrence, then*

$$(\nabla_X \overline{R})(Y,Z,U) + (\nabla_Y \overline{R})(Z,X,U) + (\nabla_Z \overline{R})(X,Y,U) = 0.$$

Proof: Let torsion tensor of special projective semi-symmetric connection is recurrent with respect to $\overline{\nabla}$ with π as 1-form of recurrence, then from the equations (3.7) and (3.19) , we have

$$\alpha(X,Z) = \frac{(c+1)^2}{4}\pi(X)\pi(Z). \tag{3.21}$$

Differentiating above equation covariantly with respect to ∇, we have

$$(\nabla_U \alpha)(X,Z) = \frac{(c+1)^2}{4}[(\nabla_U \pi)(X)\pi(Z) + \pi(X)(\nabla_U \pi)Z]$$

which due to the equation (3.19), reduces to

$$(\nabla_U \alpha)(X,Z) = \frac{(c+1)^3}{2}\pi(U)\pi(X)\pi(Z). \tag{3.22}$$

Interchanging U and X in the above equation, we have

$$(\nabla_X \alpha)(U,Z) = \frac{(c+1)^3}{2}\pi(X)\pi(U)\pi(Z). \tag{3.23}$$

In virtue of equations (3.22) and (3.23), we have

$$(\nabla_U \alpha)(X,Z) = (\nabla_X \alpha)(U,Z). \tag{3.24}$$

Now, differentiating the equation (3.4) covariantly with respect to the ∇, we have

$$(\nabla_X \overline{R})(Y,Z,U) = (\nabla_X R)(Y,Z,U) + (\nabla_X \alpha)(Y,U)Z - (\nabla_X \alpha)(Z,U)Y. \tag{3.25}$$

Writing two more equations by the cyclic permutations of X,Y and Z from above equation and adding them to equation (3.25), we get

$$(\nabla_X \overline{R})(Y,Z,U)+(\nabla_Y \overline{R})(Z,X,U)+(\nabla_Z \overline{R})(X,Y,U)=\{(\nabla_X \alpha)(Y,U)Z-(\nabla_Y \alpha)(X,U)Z\}$$
$$+\{(\nabla_Y \alpha)(Z,U)X-(\nabla_Z \alpha)(Y,U)X\}+\{(\nabla_Z \alpha)(X,U)Y-(\nabla_X \alpha)(Z,U)Y\}$$

which on using equation (3.24) ,gives

$$(\nabla_X \overline{R})(Y,Z,U)+(\nabla_Y \overline{R})(Z,X,U)+(\nabla_Z \overline{R})(X,Y,U)=0.$$

This completes the proof.

Theorem 3.6 *If the curvature tensor of special projective semi-symmetric connection vanishes and torsion tensor is recurrent with respect to* $\overline{\nabla}$ *with* π *as* 1*-form of recurrence, then manifold* M^n *satisfies the condition*

$$(\nabla_X Ric)(Y,Z)=B(X)Ric(Y,Z),$$

where $B(X)=2(c+1)\pi(X)$.

Proof: Let torsion tensor of special projective semi-symmetric connection is recurrent with respect to $\overline{\nabla}$ with π as 1-form of recurrence, then from equations (3.21) and (3.23), we have

$$(\nabla_X \alpha)(Y,Z)=B(X)\alpha(Y,Z), \tag{3.26}$$

i.e., tensor α is recurrent.

Suppose curvature tensor of special projective semi-symmetric connection is vanishes, i.e.,

$$\overline{R}(X,Y,Z)=0. \tag{3.27}$$

Now, in view of equation (3.27), the equation (3.5) gives

$$Ric(Y,Z)=(n-1)\alpha(Y,Z).$$

Differentiating above equation covariantly with respect to ∇ and using equation (3.26), we get

$$(\nabla_X Ric)(Y,Z)=B(X)Ric(Y,Z).$$

This completes the proof.

Theorem 3.7 *If the torsion tensor of special projective semi-symmetric connection in* M^n *is covariant constant with respect to Levi-Civita connection, then manifold admits a parallel vector field.*

Proof: Suppose torsion tensor of special projective semi-symmetric connection in M^n is covariant constant with respect to ∇, i.e.,

$$(\nabla_X \overline{T})(Y, Z) = 0,$$

then from the equation (2.2), we get

$$(\nabla_X \pi)(Z)Y - (\nabla_X \pi)(Y)Z = 0,$$

which on contraction, gives

$$(\nabla_X \pi)(Z) = 0. \tag{3.28}$$

Differentiating the equation (2.4) covariantly with respect to ∇, we get

$$(\nabla_X \pi)(Z) = g(Z, \nabla_X \rho). \tag{3.29}$$

From the equations (3.28) and (3.29), we have $\nabla_X \rho = 0$,

which shows that vector field ρ is parallel vector field.

References

1. A. Friedmann and J. A Schouten, Uber die geometrie der halbsymmetrischen Ubertragungen, *Math. Zeitschr.*, **21(1)**, (1924) 211-223.

2. H. N. Hayden, Subspaces of space with torsion, *Proc. London Math. Soc.*, **34**, (1932) 27-50.

3. K. Yano, On semi-symmetric metric connection, *Revue Roumaine de Math. Pure et Appliquees*, **15**, (1970) 1579-1581.

4. T. Imai, Notes on semi-symetric metric connections, *Tensor N. S.,* **24**, (1972) 293-296.

5. R. S. Mishra, and S. N. Pandey., Semi-symmetric metric connection in an almost contact manifold, *Indian J. Pure Appl. Math.*, **9(6)**, (1978) 570-580.

6. U.C. De and B. K De, On a type of semi-symmetric connection on a Riemannian manifold, *Ganita*, **47(2)**, (1996) 11-24.

7. Y. Liang., Some Properties of the Semi-symmetric Metric Connection, *J of Xiamen University (Natural Science)*, **30(1)**, (1991) 22-24.

8. P. Zhao and L. Shangguan, On semi-symmetric connection, *J. of Henan Normal University (Natural Science)*, **19(4)**, (1994) 13-16.

9. P. Zhao and H. Song, An invariant of the projective semisymmetric connection, *Chinese Quarterly J. of Math*., **17(4)**, (2001) 48-52.

10. Fu. Fengyun and P. Zhao, A property on geodesic mappings of pseudo-symmetric Riemannian manifold*s*, *Bull. Malays. Math. Sci. Soc*.(2), **33(2)**, (2010) 265-272.

11. Y. Han, H. Yun and P. Zhao, Some invariants of quarter-symmetric metric connections under the projective transformation, *Filomat*, **27(4)**, (2013) 679-691.

12. P. Zhao, Some properties of projective semi-symmetric connections, *Int. Math.Forum*, **3(7)**, (2008) 341-347.

13. R. S Mishra, *Structures on a differentiable manifolds and their applications*, Chandrama Prakashan, Allahabad, 1984.