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CR-Submanifolds of (ϵ)-Lorentzian Para-Sasakian Manifold

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Abstract: In this paper we studied with some properties of CR-Submanifolds of (ϵ)- Lorentzian para-Sasakian manifold and dealt with totally geodesic.

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1. Introduction

In 1978, Bejancu introduced the notion of CR-Submanifold of a Kaehler manifold[1]. In 1989, K. Matsumoto[2] introduced the notion of Loretzian para-Sasakian manifolds. Also Bejancu and K. L Duggal introduced (ε)-Sasakian manifolds. CR-Submanifolds of Sasakian manifold have been studied by Kobayashi[3] and other some authors. In 1985, Obina introduced a new class of of almost contact metric manifold. I. Mihai and R. Rosca[4] defined same notion independently and several authors[5,6]. In 2012, R.Prasad and V.Shrivastva[7] introduced the CR-Submanifolds of (ε)-Lorentzian para-Sasakian manifold.The present paper is organised as follows:

Section1 is introductry and in section2 we defined (ε)-Lorentzian para-Sasakian manifold. We also give some basic results in section3. In section4 we studied parallel distribution with respect to the connection on (ε)-Lorentzian para-Sasakian manifold. In section 5, we study the CR-Submanifolds with totally geodesic properties. Finally, we study the intregrability condition on CR-Submanifolds of (ε)-Lorentzian para-Sasakian manifold.

2. Preliminaries

An n dimensional differentiable manifold $\bar{M}$ is called (ε)- Lorentziaan para-Sasakian manifold if

(1.0) $\phi^2 = I + \eta(X)\xi$, $\eta(\xi) = -1$, $\eta \circ \phi = 0$

(1.1) $g(\xi,\xi) = \varepsilon$, $\eta(X) = \varepsilon g(X,\xi)$

(1.2) $g(\phi X, \phi Y) = g(X,Y) + \varepsilon\eta(X)\eta(Y)$

where X and Y are the vector fields tangent to $\bar{M}$ and ε is 1 or -1 according as ξ is space like or time like vector field.

Also in (ε)-Lorentzian para-Sasakian manifold, we have

(1.3) $(\bar{\nabla}_X \phi)Y = g(X,Y)\xi + 2\varepsilon\eta(X)\eta(Y) + \varepsilon\eta(Y)$

where $\bar{\nabla}$ denotes the operator of covariant differetiation with respect to the Lorentzian metric g on $\bar{M}$.

Further,

(1.4) $(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = 2g(X,Y)\xi + 4\varepsilon\eta(X)\eta(Y) + \varepsilon\eta(Y)X + \varepsilon\eta(X)Y$.

Let M be an m dimensional isometrically immersed submanifold of (ε)-Lorentzian para-Sasakian manifold $\bar{M}$ and denote by the same g the Lorentzian metric tensor field induced on M from that of $\bar{M}$.

Definition 1.1 *An m dimensional Riemannian submanifold M of (ε)-Lorentzian para-Sasakian manifold $\bar{M}$ is called a CR-Submanifold if ξ is tangent to M and there exits a differentiable distribution $D : x \in M \rightarrow D_x \subset T_x M$ such that*

(i) the distribution D_x is invariant under φ, that is

$$\varphi D_x \subset D_x \text{ for each } x \in M;$$

(ii) the complementary orthogonal distribution

$$D^{\perp} : x \in M \rightarrow D_x^{\perp} \subset T_x M$$

of D is anti-invariant under φ that is $\varphi D_x^{\perp} \subset T_x^{\perp} M$ *for each* $x \in M$;

where $T_x M$ and $T_x^{\perp} M$ are the tangent space and the normal space of M at x respectivly.

If dim $D_x^{\perp}$ = 0 (resp., dim D_x= 0), then the CR- Submanifold is called an invariant (resp., anti-invariant) submanifold. The distribution D (resp., $D^{\perp}$) is called the horizontal (resp., vertical) distribution. Also the pair $(D, D^{\perp})$ is called ξ –horizontal (resp., vertical) if $\xi_x \in D_x$ (resp., $\xi_x \in D^{\perp}$)[10].

For any vector field X tangent to M, we put [10]

(1.5) $X = PX + QX$

where PX and QX belong to the distribution D and $D^{\perp}$.

For any vector field normal to M, we have

(1.6) $\phi N = BN + CN$

where BN and CN denote the tangential and normal component of φN respectively.

Let $\bar{\nabla}$ (resp., ∇) be the covariant differentiation with respect to the Levia-civita connection on $\bar{M}$(resp., M). The Gauss and Weingarten formulas for M are respectively given by

(1.7) $\overline{\nabla}_X Y = \nabla_X Y + h(X, Y)$

(1.8) $\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N$

for X, Y ∈ TM and N∈ $T^{\perp}$M where h (resp., A) is second fundamental form (resp., tensor) of M in $\bar{M}$ and $\nabla^{\perp}$ denotes the normal connection.
Moreover, we have

(1.9) $g(h(X, Y), N) = g(A_N X, Y)$.

3. Some Basic Results

First we prove the following lemma:

Lemma 1.2 *Let M be a CR-Submanifold of an (ε)-Lorentzian para-Sasakian manifold M. Then*

(2.0)
$$P(\nabla_X \varphi PY) - P(\nabla_Y \varphi PX) - P(A_{\varphi QX} Y) - P(A_{\varphi QY} X) = 2g(X, Y)P\xi + \varepsilon\eta(Y)PX + \varphi P\nabla_Y X + 4\varepsilon\eta(X)\eta(Y)$$

(2.1)
$$Q(\nabla_X \varphi PY) + Q(\nabla_Y \varphi PX) - Q(A_{\varphi QY} X) - Q(A_{\varphi QX} Y) = 2g(X, Y)Q\xi + \varepsilon\eta(Y)QX + \varepsilon\eta(X)QY + 2Bh(X, Y)$$

(2.2)
$$h(X, \varphi PY) + h(Y, \varphi PX) + \nabla_X^{\perp} \varphi QY + \nabla_Y^{\perp} \varphi QX = \varphi Q\nabla_Y X + \varphi Q\nabla_X Y + 2Ch(X, Y)$$

for X,Y ∈ TM.

Proof: From (1.3),(1.5),(1.6),(1.7) and (1.8), we have

$$\nabla_X \varphi PY + h(X,\varphi PY) + \nabla_Y \varphi PX + h(Y,\varphi PX) - A_{\varphi QX} Y$$
$$-A_{\varphi QY} X + \nabla^{\perp}_X \varphi QY + \nabla^{\perp}_Y \varphi QX - \varphi P \nabla_X Y - \varphi Q \nabla_X Y$$
$$-\varphi P \nabla_Y X - \varphi Q \nabla_Y X = 2g(X,Y) P\xi + 2g(X,Y) Q\xi \tag{2.3}$$
$$+4\varepsilon\eta(X)\eta(Y) + \varepsilon\eta(Y) PX + \varepsilon\eta(X) PY$$
$$+\varepsilon\eta(Y) QX + \varepsilon\eta(X) QY + 2Bh(X,Y) + 2Ch(X,Y)$$

for any X,Y∈TM.

Now using (1.5) and equating horizontal, vertical and normal components in (2.3), we get the result.

Lemma 1.3 *Let M be a CR-Submanifold of an (ε)-Lorentzian para-Sasakian manifold* $\bar{M}$ *.Then*

$$2(\bar{\nabla}_X \varphi) Y = \nabla_X \varphi Y + h(X,\varphi Y) - \nabla_Y \varphi X - h(Y,\varphi X) - [X,Y] \tag{2.4}$$
$$+ 2g(X,Y)\xi + 4\varepsilon\eta(X)\eta(Y) + \varepsilon\{\eta(Y)X + \eta(X)Y\}$$

for any X,Y∈D.

Proof: By using Gauss formula (1.7), we get

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = \nabla_X \phi Y + h(X,\phi Y) - \nabla_Y \phi X - h(Y,\phi X). \tag{2.5}$$

Also, we have

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = (\bar{\nabla}_X \phi) Y - (\bar{\nabla}_Y \phi) X + \phi[X,Y]. \tag{2.6}$$

From (2.5) and (2.6), we get

$$(\bar{\nabla}_X \varphi) Y - (\bar{\nabla}_Y \varphi) X = \nabla_X \varphi Y + h(X,\varphi Y) - \nabla_Y \varphi X - h(Y,\varphi X) - \varphi[X,Y] \tag{2.7}$$

Also for (ε)-Lorentzian para-Sasakian manifold, we have

$$(\bar{\nabla}_X \varphi) Y + (\bar{\nabla}_Y \varphi) X = 2g(X,Y)\xi + 4\varepsilon\eta(X)\eta(Y) + \varepsilon\eta(Y) X + \varepsilon\eta(X) Y \tag{2.8}$$

Combining (2.7) and (2.8), the lemma follows.

In particular, we have the following corollary:

Corollary 1.4 *Let M be a ξ -vertical CR-Submanifold of an (ε)-Lorentzian para-Sasakian manifold $\bar{M}$, then*

(2.9)
$$2\left(\bar{\nabla}_X\varphi\right)Y=\nabla_X\varphi Y+h\left(X,\varphi Y\right)-\nabla_Y\varphi X-h\left(Y,\varphi X\right)-\left[X,Y\right]+2g\left(X,Y\right)\xi+4\varepsilon\eta\left(X\right)\eta\left(Y\right)$$

for any X,Y∈D.

Similarly, Weingarten formula (1.8), we get the following lemma:

Lemma 1.5 *Let M be a CR-Submanifold of an (ε)-Lorentzian para-sasakian manifold $\bar{M}$, then*

(3.0)
$$2\left(\bar{\nabla}_Y\varphi\right)Z=A_{\varphi Y}Z-A_{\varphi Z}Y+\nabla_Y^{\perp}\varphi Z-\nabla_Z^{\perp}\varphi Y+2g\left(Y,Z\right)\xi+4\varepsilon\eta\left(X\right)\eta\left(Y\right)+\varepsilon\left\{\eta\left(X\right)Y+\eta\left(Y\right)X\right\}$$

for any Y,Z∈$D^{\perp}$.

Corollary 1.6 *Let M be a ξ-horizontal CR-Submanifold of an (ε)-Lorentzian para-Sasakian manifold $\bar{M}$, then*

(3.1)
$$2\left(\bar{\nabla}_Y\varphi\right)Z=A_{\varphi Y}Z-A_{\varphi Z}Y+\nabla_Y^{\perp}\varphi Z-\nabla_Z^{\perp}\varphi Y+2g\left(Y,Z\right)\xi+4\varepsilon\eta\left(X\right)\eta\left(Y\right)\cdot$$

Lemma 1.7 *Let M be a CR-Submanifold of an (ε)-Lorentzian para-Sasakian manifold $\bar{M}$, then*

(3.2)
$$2\left(\bar{\nabla}_X\varphi\right)Y=2g\left(X,Y\right)\xi-A_{\varphi Y}X+\nabla_X^{\perp}\varphi Y-\nabla_Y\varphi X-h\left(Y,\varphi X\right)+4\varepsilon\eta\left(X\right)\eta\left(Y\right)-\varphi\left[X,Y\right]+\varepsilon\left\{\eta\left(X\right)Y-\eta\left(Y\right)X\right\}$$

for any X∈D and Y ∈ $D^{\perp}$.

4. Parallel Distribution

Definition 1.8 *The horizontal (resp., vertical) distribution D (resp., $D^{\perp}$) is said to be parallel*[1] *with respect to the connection ∇ on M if $\nabla_X Y\in D$ (resp., $\nabla_Z W\in D^{\perp}$) for any vector field X,Y∈D (resp., W,Z∈$D^{\perp}$).*

Now, we prove the following preposition.

Proposition 1.9 *Let M be a ξ-vertical CR-Submanifold of an (ε)-Lorentzian para-Sasakian manifold $\bar{M}$. If the horizontal distribution D is parallel, then*

(3.3) $h(X,\phi Y)=h(Y,\phi X)$ *for all X, Y∈D.*

Proof: Using parallelism of horizontal distribution D, we have

$\nabla_X \varphi Y \in D$, $\nabla_Y \varphi X \in D$ for any X,Y∈D.

From (2.1) we have

(3.4) $g(X,Y)Q\xi = Bh(X,Y)$.

Also since

(3.5) $\phi h(X,Y)=Bh(X,Y)+Ch(X,Y)$

then

(3.6) $\phi h(X,Y)=g(X,Y)Q\xi + Ch(X,Y)$ for any X,Y∈ D.

Next from (2.2)

(3.7) $h(X,\phi Y)+h(Y,\phi X)=2\phi h(X,Y)-2g(X,Y)Q\xi$ for any X, Y∈D.

Since h is symmetric, by putting X=φX∈D and Y=φY∈D in (3.6) we get

(3.8) $\phi h(\phi X,Y)-\phi h(X,\phi Y)=g(\phi X,Y)Q\xi - g(X,\phi Y)Q\xi$.

Operating φ on the both side of above equation and using φ∘ξ= 0 we have the proposition.

Now for the distribution $D^{\perp}$, we prove the following proposition:

Proposition 2.0 *Let M be a ξ-vertical CR-Submanifold of a (ε)-Lorentzian para-Sasakian manifold $\bar{M}$. If the distribution $D^{\perp}$, is parallel with respect to the connection on M, then*

(3.9) $(A_{\phi Z}Y - A_{\phi Y}Z)\in D^{\perp}$ *for any Y,Z∈ $D^{\perp}$.*

Proof: Let Y,Z ∈ $D^{\perp}$, , then using Gauss and Weingarten formula (1.7) and (1.8), we obtain

(4.0) $-A_{\phi Y}Z + A_{\phi Z}Y + \nabla_Z^{\perp}\phi Y - \nabla_Y^{\perp}\phi Z = \varepsilon\{\eta(Y)Z - \eta(Z)Y\} + \phi[Z,Y]$.

Taking inner product with X∈D in (4.0), we get

(4.1) $g(-A_{\phi Y}Z + A_{\phi Z}Y, X)=0$

which is equivalent to $\left(-A_{\phi Y}Z + A_{\phi Z}Y, X\right) \in D^{\perp}$ for any Y,Z∈ $D^{\perp}$, and this completes the proof.

5. Totally Geodesic

Definition.2.1 *A CR-Submanifold is said to be D-totally geodesic (resp., $D^{\perp}$-totally geodesic) if h(X,Z)= 0 for all X,Z∈D (resp., h(X,Z)= 0, for all X,Z∈$D^{\perp}$).*

Proposition.2.2 *Let M be a CR-Submanifold of (ε)-Lorentzian para-Sasakian manifold $\bar{M}$, then*

(i) M is D-totally geodesic if and only if $A_N X \in D^{\perp}$.

(ii) M is $D^{\perp}$-totally geodesic if and only if $A_N X \in D$.

Proof. From (1.9) and hypothesis, we get

(4.2) $\qquad g(A_N X, Y) = 0$

From (4.2), we get the result.

Conversly from (1.9), we get

(4.3) $\qquad g(h(X,Y), N) = 0$

From (4.3),we complete the proposition (i).

Similarly, we get the second.

Definition.2.3 *A CR-Submanifold is said to be mixed totally geodesic if h(X,Y)=0 for all X∈D,Z∈$D^{\perp}$.*

Lemma.2.4: *Let M be a CR-Submanifold of (ε)-Lorentzian para-Sasakian manifold M. Then M is mixed totally geodesic if and only if $A_N X \in D$ for all X∈D.*

Definition.2.5 *A CR-Submanifold of (ε)-Lorentzian para-Sasakian manifold M is called D-umbilic (resp. $D^{\perp}$-umbilic) if*

(4.4) $\qquad h(X,Y) = g(X,Y)H$

holds for all X,Y∈D (resp.X,Y∈$D^{\perp}$), where H is mean curvature vector field.

Proposition.2.6 *Let M be a D-umbilic ξ-horizontal CR-submanifold of an (ε)-Lorentzian para-Sasakian manifold $\bar{M}$, then M is D-totally geodesic.*

Proof: Let M be D-umbilic ξ-horizontal CR-Submanifold, then by putting X=Y=ξ in (4.4), we get

(4.5) $H = 0$.

Now using (4.4), we have

$h(X,Y) = 0$, which proves that M is D-totally geodesic.

6. Integrability Condition of Distribution

We calculate the Nijenhuis tensor N_φ(X, Y) on an (ε)-Lorentzian para-Sasakian manifold $\bar{M}$. For this first we prove the following lemma:

Lemma.2.7 *Let $\bar{M}$ be the an (ε)-Lorentzian para-Sasakian manifold, then*

(4.6) $$\left(\bar{\nabla}_{\phi X}\phi\right)Y = 2g(\phi X, Y)\xi + \varepsilon\eta(Y)\phi X - \eta(X)\bar{\nabla}_Y\xi + \eta\left(\bar{\nabla}_Y X\right)\xi + \phi\left(\bar{\nabla}_Y\phi\right)X$$

for any X,Y∈TM.

Proof: From the definition of (∈)-Lorentzian para-Sasakian manifold $\bar{M}$ we have

(4.7) $$\left(\bar{\nabla}_{\phi X}\phi\right)Y = 2g(\phi X, Y)\xi + \varepsilon\eta(Y)\phi X - \left(\bar{\nabla}_Y\phi\right)\phi X.$$

Also we have,

(4.8) $$\left(\bar{\nabla}_Y\phi\right)\phi X = \eta(X)\bar{\nabla}_Y\xi - \phi\left(\bar{\nabla}_Y\phi\right)X - \eta\left(\bar{\nabla}_Y X\right)\xi.$$

for any X,Y∈$T\bar{M}$.

Using (4.8) in (4.7), we get the lemma.

On (ε)-Lorentzian para-Sasakian manifold $\bar{M}$, Nijenhuis tensor is given by

(4.9) $$N_\phi(X,Y) = \left(\bar{\nabla}_{\phi X}\phi\right)(Y) - \left(\bar{\nabla}_{\phi Y}\phi\right)(X) - \phi\left(\bar{\nabla}_X\phi\right)(Y) + \phi\left(\bar{\nabla}_Y\phi\right)(X)$$

for any X,Y∈T$\bar{M}$.

From (4.6) and (4.9), we have

(5.0) $$\begin{aligned} &N_\varphi(X,Y) = 4g(\varphi X,Y)\xi - \varepsilon\eta(Y)\varphi X - 3\varepsilon\eta(X)\varphi Y - \eta(X)\bar{\nabla}_Y\xi \\ &+\eta\left(\bar{\nabla}_Y X\right)\xi + \eta(Y)\bar{\nabla}_X\xi - \eta\left(\bar{\nabla}_X Y\right)\xi + 4\varphi\left(\bar{\nabla}_Y\varphi\right)X - 8\varphi\varepsilon\eta(X)\eta(Y)\end{aligned}$$

for any X,Y∈TM.

Proposition.2.8 *Let M be a ξ-vertical CR-Submanifold of (ε)-Lorentzian para-Sasakian manifold $\bar{M}$. Then the distribution D is integrable if the following conditions are satisfied:*

$$(5.1) \qquad S(X,Z) \in D, \qquad h(X,\phi Z) = h(\phi X, Z)$$

for any X,Z∈D.

Proof: The torsion tensor

$$(5.2) \qquad S(X,Y) = N_\phi(X,Y) + 2d\eta(X,Y)\xi = N_\phi(X,Y) + 2g(\phi X, Y)\xi .$$

Thus

$$(5.3) \qquad S(X,Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + 2g(\phi X, Y)\xi$$

for any X,Y∈TM.

Suppose that the distribution D is integrable. So for X,Y∈D, Q[X,Y]=0 and η([X,Y])=0 as ξ∈$D^\perp$).

If S(X, Y)∈D, then from (5.0) and (5.3) we have

$$(5.4) \qquad 6g(\phi X, Y)\xi + 4(\phi\nabla_Y \phi X) + \phi h(Y,\phi X) + \nabla_Y X + h(X,Y)) \in D$$

from (5.4),we have

$$(5.5) \qquad 6g(\phi X, Y)Q\xi + 4(\phi Q\nabla_Y \phi X + \phi h(Y,\phi X) + Q\nabla_Y X + h(X,Y)) = 0 .$$

Replacing Y by φZ for Z∈D in (5.5), we have

$$(5.6) \qquad 6g(\phi X, \phi Z)Q\xi + 4(\phi Q\nabla_{\phi Z} \phi X + \phi h(\phi Z,\phi X) + Q\nabla_{\phi Z} X + h(\phi Z, X)) = 0$$

Interchanging X and Z for X,Z∈D in (5.6) and subtracting these relations, we get

$$(5.7) \qquad \phi Q[\phi X, \phi Z] + Q[X, \phi Z] - h(Z, \phi X) + h(X, \phi Z) = 0$$

for any X,Y∈D and the assertion follow.

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