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Study of Conharmonic Recurrent Symmetric Kaehler Manifold with Semi-Symmetric Metric Connection

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Abstract: This paper contains the study of Kaehler manifold with different approaches. We consider a Kaehler manifold with semi-symmetric metric connection and obtain conharmonic curvature tensor with respect to semi-symmetric metric connection. We study conharmonic recurrent Kaehler manifold with respect to semi-symmetric metric connection and obtain interesting results. Also, we discuss Kaehlerian conharmonic symmetric manifold with respect to a semi-symmetric metric connection.

Keywords: Kaehler manifold, semi- symmetric metric connection, Ricci-recurrent, recurrent curvature tensor, Kaehlerian conharmonic symmetric manifold.

2010 Mathematics Subject Classification: 53C10, 53C15.

1. Introduction

In 1967, O. C. Andoni[1] and in 1981, M. C. Chaki[2] studied the theory of semi-symmetric metric connection. B. B. Chaturvedi and P. N. Pandey[3,4,5] studied semi-symmetric metric connection in Kaehler and Hermitian manifold.

Let $(M^n, g), (n>2)$ be an even dimensional manifold with structure F_i^h, if F_i^h satisfies the relation

(1.1) $$F_j^i F_i^h = -\delta_j^h,$$

the manifold is called almost complex manifold. In almost complex manifold, if

(1.2) $$F_{ij} = -F_{ji} \quad (F_{ij} = g_{jk} F_i^k),$$

then the manifold is called Hermitian manifold.

In Hermitian manifold if

(1.3) $F^h_{i,j} = 0,$

then the manifold is called a Kaehler manifold i.e. in a Kaehler manifold equation (1.1), (1.2) and (1.3) hold, where F_i^h is a tensor field of type (1, 1) and $F^h_{i,j}$ is covariant derivative of F_i^h with respect to Riemannian connection.

A Friedman and J. A. Schouten[6] considered a semi-symmetric metric connection ∇ and a Riemannian connection D with coefficients Γ^h_{ij} and $\{^h_{ij}\}$ respectively. According to them if the torsion tensor T of the connection ∇ on (M^n, g), $(n > 2)$ be

(1.4) $$T^h_{ij} = \delta^h_i\, \omega_j - \delta^h_j\, \omega_i,$$

then

(1.5) $$\Gamma^h_{ij} = \left\{^h_{ij}\right\} + \delta^h_i\, \omega_j - g_{ij}\, \omega^h,$$

where $\omega^h = \omega_t\, g^{th}$, ω^h being the contravariant components of the generating vector ω_h and

(1.6) $$\nabla_j \omega_i = D_j \omega_i - \omega_i \omega_j + g_{ij}\, \omega,$$

where $\omega = \omega^h \omega_h$.

Friedman and J. A. Schouten[6] also shown that the curvature tensor with respect to semi-symmetric metric connection and Riemannian connection are related by

(1.7) $$\overline{R}_{ijkm} = R_{ijkm} - g_{im}\pi_{jk} + g_{jm}\pi_{ik} - g_{jk}\pi_{im} + g_{ik}\pi_{jm},$$

where

(1.8) $$\pi_{jk} = \nabla_j \omega_k - \omega_j \omega_k + \frac{1}{2} g_{jk}\, \omega.$$

Transvecting (1.7) by g^{mh}, we get

(1.9) $$\overline{R}^h{}_{ijk} = R^h{}_{ijk} - \delta^h_i \pi_{jk} + \delta^h_j \pi_{ik} - g_{jk}\pi^h_i + g_{ik}\pi^h_j.$$

If we take $\delta^h_i \pi_{jk} = \delta^h_j\, \pi_{ik}$ then the equation (1.9) can be written as

(1.10) $$\overline{R}^h{}_{ijk} = R^h{}_{ijk} - g_{jk}\pi^h_i + g_{ik}\pi^h_j.$$

Again transvecting (1.10) by g^{jk}, we get

(1.11) $\overline{R}^h{}_i = R^h{}_i - (n-1)\pi_i^h$.

We well known that Riemannian curvature tensor $R^h{}_{ijk}$, Rcci tensor R_{ij} and scalar curvature tensor R are defined by

(1.12) $R^h{}_{ijk} = \partial_i\{{}^{h}_{jk}\} - \partial_j\{{}^{h}_{ik}\} + \{{}^{h}_{il}\}\{{}^{l}_{jk}\} - \{{}^{h}_{jl}\}\{{}^{l}_{ik}\}$,

(1.13) $R_{ij} = R^a_{aij}$,

and

(1.14) $R = R_{ij}g^{ij}$.

In (1967) Tachibana found the following identities

(1.15) $F_i^a R_a^j = R_i^a F_a^j$,

and

(1.16) $F_i^a R_{aj} = -R_{ia}F_j^a$.

From (1.1) and (1.15), we get

(1.17) $F_i^a R_a^b F_b^j = -R_i^j$.

Transvecting (1.16) by g^{ij}, we get

(1.18) $F_i^a R_a^i = -R_a^j F_j^a$.

Equation (1.18) implies

(1.19) $F_i^a R_a^i = 0$.

The holomorphically conharmonic curvature tensor in Riemannian manifold is defined by

(1.20) $T_{ijk}^h = R_{ijk}^h + \dfrac{1}{(n-2)}(g_{ik}R_j^h - g_{jk}R_i^h)$.

Therefore the holomorphically conharmonic curvature tensor with respect to semi-symmetric metric connection is given by

(1.21) $$\overline{T}^h_{ijk} = \overline{R}^h_{ijk} + \frac{1}{(n-2)}(g_{ik}\overline{R}^h_j - g_{jk}\overline{R}^h_i),$$

where $\overline{R}^h_{ijk}$, $\overline{R}^h_j$ and $\overline{R}$ are called curvature tensor, Ricci tensor and scalar curvature tensor with respect to semi-symmetric metric connection respectively.
Using (1.10) and (1.11) in (1.21), we have

(1.22) $$\overline{T}^h_{ijk} = R^h_{ijk} - g_{jk}\pi^h_i + g_{ik}\pi^h_j + \frac{1}{(n-2)}\left[g_{ik}(R^h_j - (n-1)\pi^h_j) - g_{jk}(R^h_i - (n-1)\pi^h_i)\right]$$

From (1.17) and (1.22), we have

(1.23) $$\overline{T}^h_{ijk} = R^h_{ijk} - g_{jk}\pi^h_i + g_{ik}\pi^h_j + \frac{1}{(n-2)}(g_{ik}(-F^a_j R^b_a F^h_b - (n-1)\pi^h_j) - g_{jk}(-F^a_i R^b_a F^h_b - (n-1)\pi^h_i)).$$

Equation (1.23) implies

(1.24)
$$\overline{T}^h_{ijk} = R^h_{ijk} - \frac{R^b_a F^h_b}{(n-2)}(g_{ik}F^a_j - g_{jk}F^a_i) + \frac{1}{(n-2)}(g_{jk}\pi^h_i - g_{ik}\pi^h_j).$$

Using (1.2) in the equation (1.24), we have

(1.25) $$\overline{T}^h_{ijk} = R^h_{ijk} + \frac{2R^b_k F_{ij} F^h_b}{(n-2)} + \frac{1}{(n-2)}(g_{jk}\pi^h_i - g_{ik}\pi^h_j).$$

Thus we have:

Theorem1.1. *In a Hermitian manifold $(M^n, g), (n > 2)$ the holomorphic conharmonic curvature tensor with respect to semi-symmetric metric connection is given by (1.25)and the holomorphic conharmonic curvature with respect to semi-symmetric metric connection will be equal to Riemannian curvature tensor if and only if*

(1.26) $$F_{ij} R^b_k F^h_b = \pi^h_{[j} g_{i]k}.$$

Definition 1.2. *In a Kaehler manifold if the curvature tensor satisfies relation*

(1.27) $$R^h_{ijk,a} = \lambda_a \ R^h_{ijk}.$$

is called Kaehlerian recurrent manifold, where λ_a *is a non -zero recurrence vector*

A Kaehler manifold which satisfies the relation

(1.28) $$R_{ij,a} = \lambda_a \ R_{ij}.$$

is called Ricci- recurrent Kaehlerian manifold, where λ_a is a non -zero recurrence vector

Transvecting (1.28) by g^{hj} , we have

(1.29) $$R^h_{i,a} = \lambda_a \ R^h_i.$$

Definition 1.3. *The Kaehler manifold in which the relation*

(1.30) $$T_{\ k,a} = \lambda_a \ T^h_{ijk},$$

satisfies is called a Kaehlerian conharmonic recurrent manifold, where λ_a *a is non-zero recurrence vector.*

Taking covariant derivative of (1.24) with respect to ∇ , we have

(1.31) $$\nabla_a \overline{T}^h_{ijk} = \nabla_a R^h_{ijk} - \frac{\nabla_a R^b_l F^h_b}{(n-2)}(g_{ik}F^l_j - g_{jk}F^l_i)$$
$$+\frac{1}{(n-2)}(g_{jk}\nabla_a \pi^h_i - g_{ik}\nabla_a \pi^h_j) - \frac{R^b_l \nabla_a F^h_b}{(n-2)}(g_{ik}F^l_j - g_{jk}F^l_i)$$
$$+\frac{1}{(n-2)}(\nabla_a g_{jk}\pi^h_i - \nabla_a g_{ik}\pi^h_j) - \frac{R^b_l F^h_b}{(n-2)}(g_{ik}\nabla_a F^l_j - g_{jk}\nabla_a F^l_i).$$

By straight forward calculation we can easily get

$$(a) \quad \nabla_a R^h_{ijk} = D_a R^h_{ijk},$$
$$(b) \quad \nabla_a R^h_i = D_a R^h_i,$$
$$(c) \quad \nabla_a \pi^h_i = D_a \pi^h_i,$$

$$(d) \quad \nabla_a F^h_i = D_a F^h_i,$$
$$(e) \quad \nabla_a g_{ij} = D_a g_{ij}.$$

If we take $\pi^h_{i,a} = \lambda_a \pi^h_i$ and using (1.3), (1.29) and (a, b, c, d, e) in (1.31), we have

$$(1.32)\quad \nabla_a \overline{T}_{ijk}^{h} = \lambda_a \left(R_{ijk}^{h} - \frac{R_l^b F_b^h}{(n-2)} (g_{ik} F_j^l - g_{jk} F_i^l) + \frac{1}{(n-2)} (g_{jk} \pi_i^h - g_{ik} \pi_j^h) \right).$$

From (1.24) and (1.32), we have

$$(1.33)\qquad \nabla_a \overline{T}_{ijk}^{h} = \lambda_a \overline{T}_{ijk}^{h}.$$

Thus we have:

Theorem 1.4. *In a Kaehler manifold* $(M^n, g), (n > 2)$ *equipped with semi-symmetric metric connection if* π_i^h *be recurrent with respect to semi-symmetric metric connection then the manifold will be called Kaehlerian conharmonic recurrent manifold with respect to semi- symmetric metric connection.*

Taking covariant derivative of (1.24) with respect to ∇, we have

$$(1.34)\quad \nabla_a \overline{T}_{ijk}^{h} = \nabla_a R_{ijk}^{h} - \frac{\nabla_a R_l^b F_b^h}{(n-2)} (g_{ik} F_j^l - g_{jk} F_i^l)$$
$$+ \frac{1}{(n-2)} (g_{jk} \nabla_a \pi_i^h - g_{ik} \nabla_a \pi_j^h) - \frac{R_l^b \nabla_a F_b^h}{(n-2)} (g_{ik} F_j^l - g_{jk} F_i^l)$$
$$+ \frac{1}{(n-2)} (\nabla_a g_{jk} \pi_i^h - \nabla_a g_{ik} \pi_j^h) - \frac{R_l^b F_b^h}{(n-2)} (g_{ik} \nabla_a F_j^l - g_{jk} \nabla_a F_i^l).$$

Using (1.3) in (1.34), we have

$$(1.35)\qquad \nabla_a \overline{T}_{ijk}^{h} = \nabla_a R_{ijk}^{h} - \frac{\nabla_a R_l^b F_b^h}{(n-2)} (g_{ik} F_j^l - g_{jk} F_i^l)$$
$$+ \frac{1}{(n-2)} (g_{jk} \nabla_a \pi_i^h - g_{ik} \nabla_a \pi_j^h).$$

Transvecting (1.24) by λ_a , we have

$$(1.36)\quad \lambda_a \overline{T}_{ijk}^{h} = \lambda_a R_{ijk}^{h} - \frac{\lambda_a R_l^b F_b^h}{(n-2)} (g_{ik} F_j^l - g_{jk} F_i^l) + \frac{\lambda_a}{(n-2)} (g_{jk} \pi_i^h - g_{ik} \pi_j^h).$$

Subtracting (1.36) from (1.35), we have

(1.37) $\nabla_a\overline{T}^h_{ijk}-\lambda_a\overline{T}^h_{ijk}=(\nabla_a R^h_{ijk}-\lambda_a R^h_{ijk})-\dfrac{(\nabla_a R^b_l-\lambda_a R^b_l)F^h_b}{(n-2)}(g_{ik}F^l_j-g_{jk}F^l_i)$

$$+\frac{1}{n-2}\left(g_{jk}(\nabla_a\pi^h_i-\lambda_a\pi^h_i)-g_{ik}(\nabla_a\pi^h_j-\lambda_a\pi^h_j)\right).$$

Now let π^h_j be recurrent then theorem (1.4), we can say that $\overline{T}^h_{ijk}$ will also be recurrent i.e. if $\nabla_a\pi^h_i=\lambda_a\pi^h_i$ this implies $\nabla_a\overline{T}^h_{ijk}=\lambda_a\overline{T}^h_{ijk}$, therefore from equation (1.35) we can say that if π^h_i be recurrent then $\nabla_a\overline{R}^h_{ijk}=\lambda_a\overline{R}^h_{ijk}$.

Thus we conclude:

Theorem1.5. *In a Kaehler manifold equipped with semi-symmetric metric connection if π^h_j is recurrent with respect to connection ∇, then the manifold will be Kaehlerian recurrent manifold equipped with semi-symmetric metric connection.*

Now we propose:

Theorem1.6. *In a Kaehler manifold equipped with semi-symmetric metric connection if $\pi^h_j g_{ik}=\pi^h_i g_{jk}$* i.e $\pi^h_i g_{jk}$ *be symmetric in i and j indices then conharmonic curvature tensor with respect to semi-symmetric metric connection will be equal to conharmonic curvature tensor with respect to Riemannian connection.*

Proof. From equation (1.22), we have

(1.38) $$\overline{T}^h_{ijk}=R^h_{ijk}+\frac{1}{(n-2)}(g_{ik}R^h_j-g_{jk}R^h_i)+\frac{1}{(n-2)}(g_{jk}\pi^h_i-g_{ik}\pi^h_j).$$

If $g_{jk}\pi^h_i$ is symmetric in i and j indices then form equation (1.38), we have

(1.39) $$\overline{T}^h_{ijk}=R^h_{ijk}+\frac{1}{(n-2)}(g_{ik}R^h_j-g_{jk}R^h_i)$$

Using (1.20), we have

(1.40) $$\overline{T}^h_{ijk}=T^h_{ijk}.$$

2. Kaehlerian Conharmonic Symmetric Manifold

Definition 2.1. *A Kaehler manifold satisfying the relation*

(2.1) $$T^h_{ijk}=0.$$

is called Kaehlerian conharmonic symmetric manifold.

Therefore from theorem (1.6) we can say that if $\pi_i^h g_{jk} = \pi_j^h g_{ik}$ then $\overline{T}_{ijk}^h = T_{ijk}^h$ and if our Kaehler manifold be conharmonic symmetric then from (2.1) $T_{ijk}^h = 0$, this implies $\overline{T}_{ijk}^h = 0$.

Therefore we conclude:

Theorem 2.2. *If a Kaehler manifold equipped with semi-symmetric metric connection with condition $\pi_i^h g_{jk} = \pi_j^h g_{ik}$ be Kaehlerian conharmonic symmetric with respect to Riemannian connection then this will also be Kaehlerian conharmonic symmetric with respect to semi-symmetric metric connection.*

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