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Study on Semiconformal Kaehlerian Recurrent and Symmetric Spaces of Second Order

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Abstract: Walker[1] studied on Ruse's spaces of recurrent Curvature. Singh[2] studied on Kaehlerianspaces with recurrent Bochner Curvaturetensor. Ishii[3] introduced the notion of Conharmonic transformation under which a harmonic function transforms into a harmonic function. Negi and Rawat (1994) studied some bi-recurrent and bi-symmetric properties in a Kaehlerian space. Further, Rawat and Kumar[4] studied Weyl-Sasakian Projective and Weyl-Sasakian Conformal bi-recurrent and bi-symmetric spaces. In the present paper, we have studied and defined Semiconformal Kaehlerian recurrent and symmetric spaces of second order. Several Theorems also have been established and proved therein.

Keywords: Kaehlerian recurrent space, Kaehlerian symmetric spaces, Semiconformal curvature tensor.

2010 Mathematics Subject Classification: 53C40, 53C25, 53D10.

1. Introduction

An n-dimensional Kaehlerian space K_n is a Riemannian space which admits a tensor field F_i^h satisfying

(1.1) $$F_i^h F_j^i = -\delta_j^h$$

(1.2) $$F_{ij} = -F_{ji}, \quad \left(F_{ij} = F_i^{\ a} g_{aj}\right)$$

and

(1.3) $$F_{i,j}^h = 0,$$

where the (,) followed by an index denotes the operation of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space. The Riemannian Curvature tensor R_{ijk}^{h}, is given by

(1.4) $$R_{ijk}^{h}=\partial_i\begin{Bmatrix} h \\ jk \end{Bmatrix}-\partial_j\begin{Bmatrix} h \\ ik \end{Bmatrix}+\begin{Bmatrix} h \\ ia \end{Bmatrix}\begin{Bmatrix} a \\ jk \end{Bmatrix}-\begin{Bmatrix} h \\ ja \end{Bmatrix}\begin{Bmatrix} a \\ ik \end{Bmatrix},$$

Where $\partial_i=\frac{\partial}{\partial x^i}$ and $\{x^i\}$ denotes real local coordinates. The Ricci – tensor and scalar curvature are respectively given by

(1.5) $$R_{ij}=R_{aij}^{a} \text{ and } R=g^{ij}R_{ij}.$$

If we define a tensor

(1.6) $$S_{ij}=F_i^{\ a}R_{aj},$$

then, we have

(1.7) $$S_{ij}=S_{ji},$$

(1.8) $$F_i^{\ a}=-S_{ai}F_j^{\ a}$$

and

(1.9) $$F_i^{\ a}S_{jka}=R_{ji,k}-R_{ki,j}$$

It has been verified by Yano[5], that the metric tensor g_{ij} and the Ricci tensor denoted by R_{ij} and hybrid in i and j, therefore, we get

(1.10) $$g_{ij}=g_{sr}F_i^{\ s}F_j^{\ r}$$

and

(1.11) $$R_{ij}=R_{sr}F_i^{\ s}F_j^{\ r},$$

The Conharmonic curvature tensor H_{ijk}^{h}, is given by

(1.12) $$H^h_{ijk} = R^h_{ijk} - \frac{1}{(n-2)}\left(g_{ij}R^h_k - \delta^h_j R_{ik} + \delta^h_k R_{ij} - g_{ik}R^h_j\right)$$

and, the Semiconformal curvature tensor P^h_{ijk}, is given by

(1.13) $$P^h_{ijk} = -(n-2)bC^h_{ijk} + \left[a + (n-2)b\right]H^h_{ijk},$$

where a and b are constants not simultaneously zero and C^h_{ijk} is Conformal curvature tensor. The Conformal curvature tensor C^h_{ijk}, is given by

(1.14) $$C^h_{ijk} = R^h_{ijk} - \frac{1}{(n-2)}\left(g_{ij}R^h_k - \delta^h_j R_{ik} + \delta^h_k R_{ij} - g_{ik}R^h_j\right) + \frac{R}{(n-1)(n-2)}\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right)$$

In Particular, if $a=1$ and $b=-\frac{1}{n-2}$, then the Semiconformal curvature tensor reduces to Conformal curvature tensor whereas for $a=1$ and $b=0$, such a curvature tensor reduces into Conharmonic curvature tensor.
In view of equations (1.12) and (1.14), equation (1.13) reduces to

(1.15) $$P^h_{ijk} = a\left[R^h_{ijk} - \frac{1}{(n-2)}\left(g_{ij}R^h_k - \delta^h_j R_{ik} + \delta^h_k R_{ij} - g_{ik}R^h_j\right)\right] + b\frac{R}{(n-1)}\left(\delta^h_k g_{ij} - \delta^h_j R_{ik}\right)$$

2. Properties of Semiconformal Kaehlerian Recurrent Space of Second Order

Definition 2.1: A Kaehlerian space is said to Kaehlerian recurrent space of second order, if it satisfies.

(2.1) $$R^h_{ijk,ab} - \lambda_{ab}R^h_{ijk} = 0,$$

For some non - zero tensor λ_{ab}, and is called Kaehlerian Ricci– recurrent space of second order, if it satisfies the condition

(2.2) $$R_{ij,ab} - \lambda_{ab} R_{ij} = 0\,,$$

Multiplying the above equation by g^{ij}, we have

(2.3) $$R_{,ab} - \lambda_{ab} R = 0\,,$$

Remark 2.1: From (2.1) and (2.2), it follows that every Kaehlerian recurrent space of second order is Ricci – recurrent of second order, but the Converse is not necessarily true.

Definition 2.2: A Kaehlerian space K_n satisfying the condition

(2.4) $$H^h_{ijk,ab} - \lambda_{ab} H^h_{ijk} = 0\,,$$

For some non-zero tensor λ_{ab}, will be called Kaehlerian recurrent space with Conharmonic curvature tensor of second order.

Definition 2.3: A Kaehlerian space K_n satisfying the condition

(2.5) $$P^h_{ijk,ab} - \lambda_{ab} P^h_{ijk} = 0\,,$$

For some non - zero tensor λ_{ab}, is said to be Semiconformal Kaehlerian Recurrent space of second order.

Definition 2.4: A Kaehlerian space K_n satisfying the Condition

(2.6) $$C^h_{ijk,ab} - \lambda_{ab} C^h_{ijk} = 0\,,$$

For some non - zero tensor λ_{ab}, is said to be Kaehlerian recurrent space with Conformal curvature tensor of second order.

Now, we have the following theorems:

Theorem 2.1: *If a Kaehlerian space satisfies any two of the following properties*

(i) The space is Kaehlerian recurrent space of second order,

(ii) The space is Kaehlerian Ricci-recurrent space of second order,

(iii) The space is Kaehlerian recurrent space with Semiconformal curvature tensor of second order,then it must also satisfy the third.

***Proof*:** Differentiating equation (1.15) covariantly w. r. t. x^a, again differentiate the result thus obtained covariantly w. r. t. x^b, we have

(2.7)
$$P^h_{ijk,ab} = a\left[R^h_{ijk,ab} - \frac{1}{(n-2)}\left(g_{ij}R^h_{k,ab} - \delta^h_j R_{ik,ab} + \delta^h_k R_{ij,ab} - g_{ik}R^h_{j,ab}\right)\right] + b\frac{R_{,ab}}{(n-1)}\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right)$$

Multiplying (1.15) by λ_{ab} and subtracting the result thus obtained from (2.7), we have

(2.8)
$$P^h_{ijk,ab} - \lambda_{ab}P^h_{ijk} = a\begin{bmatrix}\left(R^h_{ijk,ab} - \lambda_{ab}R^h_{ijk}\right) - \frac{1}{(n-2)}\left(R^h_{k,ab} - \lambda_{ab}R^h_k\right)g_{ij} \\ -\left(R_{ik,ab} - \lambda_{ab}R_{ik}\right)\delta^h_j + \left(R_{ij,ab} - \lambda_{ab}R_{ij}\right)\delta^h_k\end{bmatrix} + b\frac{\left(R_{,ab} - \lambda_{ab}R\right)}{(n-1)}\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right)$$

The statement of the above theorem follows in view of equations (2.1), (2.2), (2.3),(2.5) and (2.8).

Theorem 2.2: *If a Kaehlerian space satisfies any two of the following properties then space is Kaehlerian recurrent space with Conharmonic curvature tensor of second order, the space is Kaehlerian recurrent space with Conformal curvature tensor of second order, the space is Kaehlerian recurrent space with Semiconformal curvature tensor of second order, then it must also satisfy the third.*

Proof: Differentiating (1.13) covariantly w. r. t. x^a, again differentiate the result thus obtained covariantly w .r. t. x^b, we have

(2.9)
$$P^h_{ijk,ab} = (n-2)bC^h_{ijk,ab} + \left[a + (n-2)b\right]H^h_{ijk,ab}\,.$$

Multiplying (1.13) by λ_{ab} and subtracting from (2.9), we have

$$P^h_{ijk,ab} - \lambda_{ab} P^h_{ijk} = -(n-2) b \left(C^h_{ijk,ab} - \lambda_{ab} C^h_{ijk} \right) + \left[a + (n-2) b \right] \left(H^h_{ijk,ab} - \lambda_{ab} H^h_{ijk} \right) \tag{2.10}$$

The statement of the above theorem follows in view of equations (2.4), (2.6) and (2.10).

Theorems 2.3: *The necessary and sufficient condition for a Kaehlerian recurrent space with semiconformal curvature tensor of second order to be Kaehlerian recurrent space of second order is that the space be Ricci-recurrent space of second order.*

Proof: Let the Kaehlerian recurrent space with Semiconformal curvature tensor of second order be Kaehlerian recurrent space of second order, so that equations (2.1) and (2.5) are satisfied and equation (2.8), in view of equation (2.1) and (2.5) reduces to

$$-\frac{a}{n-2} \begin{bmatrix} \left(R^h_{k,ab} - \lambda_{ab} R^h_k \right) g_{ij} - \left(R_{ik,ab} - \lambda_{ab} R_{ik} \right) \delta^h_j + \left(R_{ij,ab} - \lambda_{ab} R_{ij} \right) \delta^h_k \\ -\left(R^h_{i,ab} - \lambda_{ab} R^h_j \right) g_{ik} \end{bmatrix}$$

$$+ b \frac{\left(R_{,ab} - \lambda_{ab} R \right)}{(n-1)} \left(\delta^h_k g_{ij} - \delta^h_j g_{ik} \right) = 0 \, ,$$

Or,

$$a(n-1) \begin{bmatrix} \left(R^h_{k,ab} - \lambda_{ab} R^h_k \right) g_{ij} - \left(R_{ik,ab} - \lambda_{ab} R_{ik} \right) \delta^h_j + \left(R_{ij,ab} - \lambda_{ab} R_{ij} \right) \delta^h_k \\ -\left(R^h_{i,ab} - \lambda_{ab} R^h_j \right) g_{ik} \end{bmatrix}$$

$$+ b(n-2) \left(R_{,ab} - \lambda_{ab} R \right) \left(\delta^h_k g_{ij} - \delta^h_j g_{ik} \right) = 0 \, ,$$

which after further calculation and simplification shows that the space is Ricci-recurrent space of second order.

Conversely, let Kaehlerian recurrent space with semiconformal curvature tensor of second order be Ricci-recurrent space of second order, so that equation (2.2) and (2.3) are satisfied andequation(2.8), in view of equations (2.2), (2.3) and (2.5), reduces to

$$R^h_{ijk,ab} - \lambda_{ab} R^h_{ijk} = 0 \, ,$$

which shows that the space is Kaehlerian recurrent space of second order.

This Completes the proof of the theorem.

3. Properties of Semiconformal Kaehlerian Symmetric Space of Second Order

Definition 3.1: A Kaehlerian space is said to be Kaehlerian symmetric space of second order, if it satisfies.

(3.1) $$R^h_{ijk,ab}=0 \quad \text{or, equivalently} \; R_{ijkl,ab}=0,$$

and will be called Kaehlerian Ricci-symmetric space of second order, if it satisfies.

(3.2) $$R_{ij,ab}=0$$

Multiplying equation (3.2) by g^{ij}, we have

(3.3) $$R_{,ab}=0 .$$

Remark 3.1: From (3.1) and (3.2), it fallows that every Kaehlerian symmetric space of second order is Ricci-symmetric space of second order, but the Converse is not necessarily true.

Definition 3.2: A Kaehlerian space K_n satisfying the condition

(3.4) $$H^h_{ijk,ab}=0 \quad \text{or, equivalently} \; H_{ijkl,ab}=0,$$

will be called a Kaehlerian symmetric space with Conharmonic curvature tensor of second order.

Definition 3.3: A Kaehlerian space K_n satisfying the condition

(3.5) $$P^h_{ijk,ab}=0 \quad \text{or, equivalently} \; P_{ijkl,ab}=0,$$

will be called a Semiconformal Kaehlerian symmetric space of second order.

Definition 3.4: A Kaehlerian space K_n satisfying the condition

(3.6) $$C^{h}_{ijk,ab}=0 \quad \text{or, equivalently } C_{ijkl,ab}=0,$$

will be called a Kaehlerian symmetric space with Conformal curvature tensor of second order.

Now, we have the following theorems:

Theorem 3.1: *If a Kaehlerian space satisfies any two of the following properties*

(i) The space is Kaehlerian symmetric space of second order,

(ii) The space is Kaehlerian Ricci-symmetric space of second order,

(iii) The space is Kaehlerian symmetric space with Semiconformal curvature tensor of second order, then it must also satisfy the third.

Proof: A Kaehlerian symmetric space of second order satisfied the relation (3.1) and Kaehlerian Ricci-symmetric space of second order and Kaehlerian symmetric space with Semiconformal curvature tensor of second order characterized by (3.2) and (3.5) respectively. Therefore, the statement of the above theorem follows in view of equations (3.1), (3.2), (3.5) and (2.7).

Theorem 3.2: *If a Kaehlerian space satisfies any two of the following properties*

(i) The space is Kaehlerian symmetric space with Conharmoniccurvature tensor of second order,

(ii) The space is Kaehlerians symmetric space with Conformal curvature tensor of second order,

(iii) The space is Kaehlerian symmetric space with Semiconformal curvature tensor of second order, then it must also satisfy the third.

Proof: Kaehlerian symmetric space with Conharmonic curvature tensor of second order , Kaehlerian symmetric space with Semiconformal curvature tensor of second order and Kaehlerian symmetric space with Conformal curvature tensor of second order are characterized by (3.4), (3.5) and (3.6) respectively. Therefore, the statement of the above theorem follows in view of equations (3.4), (3.5), (3.6) and (2.9).

Theorem (3.3): *The necessary and sufficient condition for a Kaehlerian symmetric space with Semiconformal curvature tensor of second order to be Kaehlerian symmetric space of second order is that the space be Ricci symmetric space of second order.*

Proof: Kaehlerian symmetric space of second order and Kaehlerian symmetric space with Semi-conformal curvature tensor of second order characterized by the equation (3.1) and (3.5) respectively. The statement of the above theorem follows in view of equations (3.1), (3.2), (3.3), (3.5) and (2.7).

Conversely, If Ricci symmetric space of second order and Kaehlerian symmetric space with Semiconformal curvature tensor of second order given by the equations (3.2) and (3.5) respectively.Therefore, by using equations (3.2), (3.3) and (3.5) in the equation (2.7), we have

$$R^{h}_{ijk,ab} = 0 \,,$$

which shows that the space is Kaehlerian symmetric space of second order.

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