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On Hypersurface of a Finsler Space with a Special Metric

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Abstract: In the present paper, we find necessary and sufficient conditions for a hypersurface of a Finsler space with a special metric $L = \frac{\alpha^2+\beta^2}{\alpha+\beta}$, where $\alpha = (a_{ij}(x)y^i y^j)^{1/2}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a differential $1-$form on a smooth manifold, to be hyperplane of the first kind, the second kind and the third kind.

Keywords: Finsler space, Riemannian metric, $(\alpha,\beta)-$metric, hypersurface, hyperplane.

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1. Introduction

In 1972, Makoto Matsumoto[1] proposed the concept of $(\alpha,\beta)-$metric, where $\alpha = (a_{ij}(x)y^i y^j)^{1/2}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a differential $1-$form on a smooth manifold. He obtained induced and intrinsic Finsler connections[2] of a hypersurface of the Finsler space in 1985. M. Hashiguchi and Y. Ichijyo[3] in 1975 and C. Shibata[4] in 1984 studied Finsler spaces with different $(\alpha,\beta)-$metrics. In 1992, Makoto Matsumoto[5] also worked on the theory of Finsler spaces with $(\alpha,\beta)-$metric and obtained many important and interesting results. In 1980, H. Wosoughi[6] studied the theory of hypersurface of special Finsler space with an exponential $(\alpha,\beta)-$metric. L. Y. Lee, H. Y. Park and Y. D. Lee[7] also studied the theory of hypersurface of a special Finsler space with a metric $\alpha + \frac{\beta^2}{\alpha}$. In 2008, M. K. Gupta and P. N. Pandey[8] studied the hypersurface of the Finsler space equipped with a Randers conformal metric. M. K. Gupta, Abhay Singh and P. N. Pandey[9] worked on the hypersurface of a Finsler space with a special metric $\frac{\alpha^2}{\alpha-\beta}+\beta$ and obtained certain geometrical properties of the hypersurface of the Finsler space in 2013.

In this paper, we study the hypersurface of a special Finsler space F_n equipped with the metric function $L=(\alpha^2+\beta^2)/(\alpha+\beta)$ and obtain necessary and sufficient conditions for the hypersurface to be a hyperplane of the first kind, the second kind and the third kind. We use the notations of the monograph of Makoto Matsumoto[10].

2. Preliminaries

Let $F_n=(M_n,L)$ be an $n-$dimensional Finsler space on a smooth manifold M_n equipped with the fundamental metric function

(2.1) $$L=\frac{\alpha^2+\beta^2}{\alpha+\beta}=\alpha+\beta-\frac{2\alpha\beta}{\alpha+\beta},$$

where $\alpha=(a_{ij}(x)y^i y^j)^{1/2}$ be a Riemannian metric and $\beta=b_i(x)y^i$ be a differential 1-form on M_n.

Differentiating (2.1) partially with respect to α and β, we get

(2.2) $$\begin{cases} L_\alpha=\dfrac{\alpha^2-\beta^2+2\alpha\beta}{(\alpha+\beta)^2}, \quad L_\beta=\dfrac{\beta^2-\alpha^2+2\alpha\beta}{(\alpha+\beta)^2} \\ L_{\alpha\alpha}=\dfrac{4\beta^2}{(\alpha+\beta)^3}, \quad L_{\beta\beta}=\dfrac{4\alpha^2}{(\alpha+\beta)^3}, \quad L_{\alpha\beta}=\dfrac{-4\alpha\beta}{(\alpha+\beta)^3}, \end{cases}$$

where

(2.3) $$L_\alpha=\frac{\partial L}{\partial\alpha},\ L_\beta=\frac{\partial L}{\partial\beta},\ L_{\alpha\alpha}=\frac{\partial^2 L}{\partial\alpha^2},\ L_{\beta\beta}=\frac{\partial^2 L}{\partial\beta^2},\ L_{\alpha\beta}=\frac{\partial^2 L}{\partial\alpha\partial\beta}.$$

The normalized element of support $l_i=\dot{\partial}_i L$ is given by

(2.4) $$l_i=\alpha^{-1}L_\alpha y_i+L_\beta b_i,$$

where $y_i = a_{ij} y^j$.

(2.5) $$h_{ij} = p_0 a_{ij} + q_0 b_i b_j + q_{-1}(b_i y_j + b_j y_i) + q_{-2} y_i y_j,$$

where p_0 , q_0, q_{-1} and q_{-2} are given by

(2.6) $$\begin{cases} p_0 = LL_\alpha \alpha^{-1} = \dfrac{\alpha^4 - \beta^4 + 2\alpha^3\beta + 2\alpha\beta^3}{\alpha(\alpha+\beta)^3}, \\ q_0 = LL_{\beta\beta} = \dfrac{4\alpha^4 + 4\alpha^2\beta^2}{(\alpha+\beta)^4}, \\ q_{-1} = LL_{\alpha\beta}\alpha^{-1} = -\dfrac{4\alpha^2\beta + 4\beta^3}{(\alpha+\beta)^4}, \\ q_{-2} = L\alpha^{-2}(L_{\alpha\alpha} - L_\alpha \alpha^{-1}) = \dfrac{(\alpha^2+\beta^2)(\beta^3 - \alpha^3 + 3\alpha\beta^2 - 3\alpha^2\beta)}{\alpha^3(\alpha+\beta)^4}. \end{cases}$$

The fundamental metric tensor $g_{ij} = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j L^2$ of F_n is given by

(2.7) $$g_{ij} = p_0 a_{ij} + d_0 b_i b_j + p_{-1}(b_i y_j + b_j y_i) + p_{-2} y_i y_j,$$

where d_0, p_{-1} and p_{-2} are given by

(2.8) $$\begin{cases} d_0 = q_0 + L_\beta^2 = \dfrac{5\alpha^4 + \beta^4 + 6\alpha^2\beta^2 + 4\alpha\beta^3 - 4\alpha^3\beta}{(\alpha+\beta)^4}, \\ p_{-1} = q_{-1} + \dfrac{1}{L} p_0 L_\beta = -\dfrac{\alpha^4 + \beta^4 + 4\alpha^3\beta + 4\alpha\beta^3 - 6\alpha^2\beta^2}{\alpha(\alpha+\beta)^4}, \\ p_{-2} = q_{-2} + p_0^2 L^{-2} = \dfrac{\beta^5 + 4\alpha\beta^4 - 6\alpha^2\beta^3 + 4\alpha^3\beta^2 + 4\alpha^4\beta}{\alpha^3(\alpha+\beta)^4}. \end{cases}$$

The inverse metric tensor g^{ij} of g_{ij} is given by

(2.9) $$g^{ij} = p_0^{-1} a^{ij} - s_0 b^i b^j - s_{-1}(b^i y^j + b^j y^i) - s_{-2} y^i y^j.$$

Here b^i, s_0, s_{-1} and s_{-2} are given as

$$(2.10)\qquad \begin{cases} b^i = a^{ij} b_j, \\ s_0 = \dfrac{p_0 d_0 + (d_0 p_{-2} - p_{-1}^2)\alpha^2}{\xi p_0}, \\ s_{-1} = \dfrac{p_0 p_{-1} + (d_0 p_{-2} - p_{-1}^2)\beta^2}{\xi p_0}, \\ s_{-2} = \dfrac{p_0 p_{-2} + (d_0 p_{-2} - p_{-1}^2) b^2}{\xi p_0}, \end{cases}$$

where $b^2 = a_{ij} b^i b^j$ and $\xi = p_0(p_0 + d_0 b^2 + p_{-1}\beta) + (d_0 p_{-2} - p_{-1}^2)(\alpha^2 b^2 - \beta^2)$.

The components of the Cartan tensor $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ of the Finsler space F_n is given by

$$(2.11)\qquad 2p_0 C_{ijk} = p_{-1}(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \gamma_1 m_i m_j m_k$$

where

$$(2.12)\qquad \gamma_1 = p_0 \frac{\partial d_0}{\partial \beta} - 3 p_{-1} q_0, \qquad m_i = b_i - \frac{1}{\alpha^2}\beta y_i.$$

Let $\left\{ {}^{\,i}_{jk} \right\}$ denote the components of Christoffel symbol of the associated $n-$dimensional Riemannian space R^n and ∇_k be the covariant differential operator with respect to x^k relative to the connection $\left\{ {}^{\,i}_{jk} \right\}$.

Let us consider the following tensors

$$(2.13)\qquad 2E_{ij} = b_{ij} + b_{ji}, \qquad 2F_{ij} = b_{ij} - b_{ji},$$

where $b_{ij} = \nabla_j b_i$. If the Cartan connection of the space F_n be $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$, then the difference tensor $D^i_{jk} = F^i_{jk} - \left\{ {}^{\,i}_{jk} \right\}$ is given by

$$(2.14)\qquad \begin{aligned} D^i_{jk} = {} & B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} \\ & - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} \\ & + \lambda^s (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}), \end{aligned}$$

where

(2.15)
$$\begin{cases} B_i = d_0 b_i + p_{-1} y_i, \quad B^i = g^{ij} B_j, \quad F_i^k = g^{kj} F_{ji}, \\ B_{ij} = \dfrac{p_{-1}(a_{ij} - \alpha^{-2} y_i y_j) + (\partial d_0 / \partial \beta) m_i m_j}{2}, \\ A_j^m = B_j^m E_{00} + B^m E_{j0} + B_j F_0^m + B_0 F_j^m, \quad B_j^m = g^{mi} B_{ij}, \\ \lambda^i = B^i E_{00} + 2 B_0 F_0^i. \end{cases}$$

The suffix '0' indicates the transvection by y^i except for the quantities p_0, d_0, q_0 and s_0.

3. Induced Cartan Tensor

Let M_{n-1} be the hypersurface of the underlying manifold M_n and suppose that it is represented parametrically by the equations

(3.1) $$x^i = x^i(u^\alpha), \qquad i = 1, 2, ...n \text{ and } \alpha = 1, 2, ...\mathrm{n}-1,$$

where u^α form a coordinate system of M_{n-1}.

Let $B_\alpha^i = \dfrac{\partial x^i}{\partial u^\alpha}$ be the projection factors[11] and supposed that the matrix $\| B_\alpha^i \|$ of the projection factors is of rank $(n-1)$. If y^i, the supporting element, is assumed to be taken tangential to M_{n-1} at a point $u = (u^\alpha)$ of M_{n-1} then it may be written in terms of the projection factors such as $y^i = B_\alpha^i(u)\, w^\alpha$. Here w^α is a supporting element of M_{n-1} at the point u^α. Hence the function $\underline{L}(u, w) = L(x(u), y(u, w))$ induces a metric function on M_{n-1}. Therefore, we obtain an $(n-1)-$dimensional Finsler space $F_{n-1} = (M_{n-1}, \underline{L})$.

In the space F_{n-1}, the metric tensor $g_{\alpha\beta}$ and the Cartan tensor $C_{\alpha\beta\gamma}$ are given by

(3.2) $$g_{\alpha\beta} = g_{ij} B_\alpha^i B_\beta^j, \qquad C_{\alpha\beta\gamma} = C_{ijk} B_\alpha^i B_\beta^j B_\gamma^k.$$

A unit normal vector $N^i(u, w)$ at point $u = (u^\alpha)$ is defined by

(3.3)
$$\begin{cases} g_{ij}(x(u), y(u, w)) B_\alpha^i N^j = 0, \\ g_{ij}(x(u), y(u, w)) N^i N^j = 1. \end{cases}$$

The angular metric tensor $h_{ij} = g_{ij} - l_i l_j$ satisfy the conditions

$$(3.4) \qquad \begin{cases} h_{\alpha\beta} = h_{ij} B^i_\alpha B^j_\beta, \\ h_{ij} B^i_\alpha N^j = 0, \\ h_{ij} N^i N^j = 1. \end{cases}$$

The inverse projection factors $B^\alpha_i(u,w)$ of $B^i_\alpha(u,w)$ are defined by

$$(3.5) \qquad B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta,$$

where $g^{\alpha\beta}$ is the inverse metric tensor of $g_{\alpha\beta}$ of F_{n-1}.

From (3.3) and (3.5), we have

$$(3.6) \qquad B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^i_\alpha N_i = 0, \quad N^i N_i = 1$$

and

$$(3.7) \qquad B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j.$$

For the induced Cartan connection $IC\Gamma = (F^\alpha_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ of the hypersurface F_{n-1}, the second fundamental $h-$tensor $H_{\alpha\beta}$ and the normal curvature vector H_α are given as

$$(3.8) \qquad H_{\alpha\beta} = N_i (B^i_{\alpha\beta} + F^i_{jk} B^j_\alpha B^k_\beta) + M_\alpha H_\beta$$

and

$$(3.9) \qquad H_\alpha = N_i (B^i_{0\alpha} + G^i_j B^j_\alpha),$$

where

$$(3.10) \qquad \begin{cases} M_\alpha = C_{ijk} B^i_\alpha N^j N^k \\ B^i_{\alpha\beta} = \dfrac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} \quad \text{and} \quad B^i_{0\alpha} = B^i_{\beta\alpha} w^\beta. \end{cases}$$

From (3.8) it is clear that the second fundamental $h-$tensor $H_{\alpha\beta}$ is not symmetric. Hence

$$(3.11) \qquad H_{\alpha\beta} - H_{\beta\alpha} = M_\alpha H_\beta - M_\beta H_\alpha.$$

Equations (3.8) and (3.9) yield

(3.12) $$H_{0\alpha} = H_{\beta\alpha} w^{\beta} = H_{\alpha}, \quad H_{\alpha 0} = H_{\alpha\beta} w^{\beta} = H_{\alpha} + M_{\alpha} H_{0}.$$

The second fundamental $v-$tensor $M_{\alpha\beta}$ is defined by

(3.13) $$M_{\alpha\beta} = C_{ijk} B^{i}_{\alpha} B^{j}_{\beta} N^{k}.$$

The relative $h-$ and $v-$covariant differentiation of the projection factor $B^{i}_{\alpha}(u,w)$ and the unit normal vector $N^{i}(u,w)$ with respect to the induced Cartan connection $IC\Gamma$ are given by

(3.14) $$B^{i}_{\alpha|\beta} = H_{\alpha\beta} N^{i}, \quad B^{i}_{\alpha}|_{\beta} = M_{\alpha\beta} N^{i}$$

and

(3.15) $$N^{i}_{|\beta} = -H_{\alpha\beta} B^{\alpha}_{i} g^{ji}, \quad N^{i}|_{\beta} = -M_{\alpha\beta} B^{\alpha}_{i} g^{ji}.$$

Let us assume that $X_{i}(x,y)$ be a contra-variant vector field in the Finsler space F_{n}. The relative $h-$ and $v-$covariant differentiation for $X_{i}(x,y)$ are given by

(3.16) $$X_{i|\beta} = X_{i|j} B^{j}_{\beta} + X_{i|j} N^{j} H_{\beta}, \quad X_{i}|_{\beta} = X_{i}|_{j}\, B^{j}_{\beta}.$$

Makato Matsumoto[2] studied the different types of hyperplanes and obtained the following characteristic conditions:

Lemma 3.1. *A hypersurface* F_{n-1} *is a hyperplane of the first kind if and only if* $H_{\alpha} = 0$ *or equivalently* $H_{0} = 0$.

Lemma 3.2. *A hypersurface* F_{n-1} *is a hyperplane of the second kind if and only if* $H_{\alpha\beta} = 0$.

Lemma 3.3. *A hypersurface* F_{n-1} *is a hyperplane of the third kind if and only if* $H_{\alpha\beta} = 0$ *&* $M_{\alpha\beta} = 0$

4. Hypersurface $F_{n-1}(c)$ of the Finsler Space F_n with the Metric $L=\dfrac{\alpha^2+\beta^2}{\alpha+\beta}=\alpha+\beta-\dfrac{2\alpha\beta}{\alpha+\beta}$

Let us consider a special metric $L=\dfrac{\alpha^2+\beta^2}{\alpha+\beta}$ such that $b_i(x)=\dfrac{\partial b}{\partial x^i}$ for a scalar function $b(x)$. Let the hypersurface $F_{n-1}(c)$ be given by the equation $b(x)=c$, where c is a constant. From the parametric equation $x^i=x^i(u^\alpha)$ of the hypersurface $F^{n-1}(c)$, we obtain $b_iB^i_\alpha=0$. This implies that $b_i(x)$ are covariant components of a normal vector field of $F_{n-1}(c)$. So, along the hypersurface $F_{n-1}(c)$, we have

(4.1) $$b_iB^i_\alpha=0,\qquad b_iy^i=0.$$

Therefore in general, the induced metric $\underline{L}(u,w)$ of $F_{n-1}(c)$ is given by

(4.2) $$\underline{L}(u,w)=\sqrt{a_{\alpha\beta}(u)w^\alpha w^\beta}\,,$$

where $a_{\alpha\beta}(u)=a_{ij}(x(u),y(u,w))B^i_\alpha B^j_\beta$. Clearly (4.2) is a Riemannian metric.

At a point of the hypersurface $F^{n-1}(c)$, from (2.6), (2.8) and (2.10), we obtain

(4.3) $$\begin{cases} p_0=1,\ q_0=4,\ q_{-1}=0,\ q_{-2}=-\dfrac{1}{\alpha^2}, \\ d_0=5,\ p_{-1}=-\dfrac{1}{\alpha},\ p_{-2}=0,\ \xi=1+4b^2, \\ s_0=\dfrac{4}{(1+4b^2)},\ s_{-1}=-\dfrac{1}{\alpha(1+4b^2)},\ s_{-2}=-\dfrac{b^2}{\alpha^2(1+4b^2)}. \end{cases}$$

From (2.9) and (4.3), the inverse metric tensor g^{ij} is given by

(4.4) $$g^{ij}=a^{ij}-\frac{4}{(1+4b^2)}b^ib^j+\frac{1}{\alpha(1+4b^2)}$$
$$(b^iy^j+b^jy^i)+\frac{b^2}{\alpha^2(1+4b^2)}y^iy^j.$$

Along the hypersurface $F_{n-1}(c)$, (4.1) and (4.4) yield

(4.5) $$g^{ij}b_i b_j = \frac{b^2}{(1+4b^2)},$$

which gives

(4.6) $$b_i(x(u)) = \left(\frac{b}{\sqrt{1+4b^2}}\right)N_i,$$

where b is the length of the vector b^i.

From (4.4) and (4.5), we obtain

(4.7) $$b^i = a^{ij}b_j = b\left(\sqrt{(1+4b^2)}\right)N^i - \frac{b^2}{\alpha}y^i.$$

This leads to

Theorem 4.1: *Let $F_n = (M_n, L)$ be an $n-$dimensional Finsler space with the fundamental metric $L = \frac{\alpha^2+\beta^2}{\alpha+\beta}$ on a smooth manifold M_n such that the vector b_i is a gradient vector, i.e. $b_i(x) = \partial_i b$. If $F_{n-1}(c)$ be a hypersurface of the space F_n given by the equation $b(x) = c$ (constant), then the induced metric $\underline{L}(u,w)$ of $F_{n-1}(c)$ is a Riemannian metric given by (4.2) and the scalar function $b(x)$ is given by (4.6) and (4.7).*

Along the hypersurface $F_{n-1}(c)$, the angular metric tensor h_{ij} and the metric tensor g_{ij} are given by

(4.8) $$h_{ij} = a_{ij} + 4b_i b_j - \frac{1}{\alpha^2}y_i y_j$$

and

(4.9) $$g_{ij} = a_{ij} + 5b_i b_j - \frac{1}{\alpha}(b_i y_j + b_j y_i),$$

respectively.

Let $h_{\alpha\beta}^{(a)}$ be the angular metric tensor corresponding to the Riemannian metric $a_{ij}(x)$. Using (3.4), (4.1) and (4.8), we get

(4.10) $$h_{\alpha\beta} = h_{\alpha\beta}^{(a)}.$$

From (2.8), we obtain

(4.11) $$\frac{\partial d_0}{\partial\beta} = \frac{24\alpha^3\beta - 24\alpha^4}{(\alpha+\beta)^5}.$$

Along the hypersurface $F_{n-1}(c)$, (4.11) gives

(4.12) $$\frac{\partial d_0}{\partial\beta} = \frac{-24}{\alpha}.$$

Therefore (2.12) gives us

(4.13) $$\gamma_1 = -\frac{12}{\alpha}, \qquad m_i = b_i.$$

In view of (4.12) and (4.13), the Cartan tensor C_{ijk} becomes

(4.14) $$C_{ijk} = -\frac{1}{2\alpha}(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j) - \frac{6}{\alpha}b_ib_jb_k.$$

From (3.4), (3.13), (4.1) and (4.14), the second fundamental $v-$tensor $M_{\alpha\beta}$ can be written as

(4.15) $$M_{\alpha\beta} = -\frac{1}{2\alpha}\left(\frac{b}{\sqrt{1+4b^2}}\right)h_{\alpha\beta}.$$

Using (3.4), (3.13), (4.1) and (4.14) in (3.10), we get

(4.16) $$M_{\alpha} = 0.$$

Thus, (3.11) implies

(4.17) $$H_{\alpha\beta} = H_{\beta\alpha}.$$

Therefore, we conclude

Theorem 4.2: *The second fundamental* $v-$ *tensor* $M_{\alpha\beta}$ *of a hypersurface* F_{n-1} *of the Finsler space* F_n *with the fundamental metric (2.1) is given by (4.15) and the second fundamental* $h-$ *tensor* $H_{\alpha\beta}$ *is symmetric.*

Taking the covariant derivative of (4.1) with respect to the induced connection, we get

(4.18) $$b_{i|\beta}B^i_\alpha + b_i B^i_{\alpha|\beta} = 0.$$

Using (3.16) for the vector b_i, we get

(4.19) $$b_{i|\beta} = b_{i|j}B^j_\beta + b_i|_j\ N^j H_\beta.$$

From (4.18), (4.19) and using the fact $B^i_{\alpha|\beta} = H_{\alpha\beta}N^i$, we obtain

(4.20) $$b_{i|j}B^i_\alpha B^j_\beta + b_i|_j\ B^i_\alpha N^j H_\beta + b_i N^i H_{\alpha\beta} = 0.$$

Since $b_i|_j = -b_r C^r_{ij}$, using (3.10), (4.6) and (4.16) in (4.20), we get

(4.21) $$\left(\frac{b}{\sqrt{1+4b^2}}\right)H_{\alpha\beta} + b_{i|j}B^i_\alpha B^j_\beta = 0.$$

Since $H_{\alpha\beta}$ is symmetric, (4.21) implies that $b_{i|j}$ is also symmetric. Transvecting (4.21) with w^β, we get

(4.22) $$\left(\frac{b}{\sqrt{1+4b^2}}\right)H_\alpha + b_{i|j}B^i_\alpha y^j = 0.$$

Again transvecting (4.22) with w^α, we obtain

(4.23) $$\left(\frac{b}{\sqrt{1+4b^2}}\right)H_0 + b_{i|j}y^i y^j = 0\,.$$

In view of the Lemma 3.1 and (4.23), hypersurface F_{n-1} is the hyperplane of first kind if and only if the condition

(4.24) $$b_{i|j}y^i y^j = 0$$

holds.

Since $b_i(x)$ is a gradient vector, (2.13) reduces to

(4.25) $$E_{ij} = b_{ij}, \quad F_{ij} = 0.$$

From (2.14) and (4.25), we have

(4.26) $$\begin{aligned} D^i_{jk} = {} & B^i b_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} \\ & - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} \\ & + \lambda^s (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}). \end{aligned}$$

In view of (4.3) and (4.4), (2.15) becomes

(4.27) $$\begin{cases} B_i = 5b_i - \dfrac{1}{\alpha} y_i, \quad B^i = \dfrac{5}{(1+4b^2)} b^i - \dfrac{1}{\alpha(1+4b^2)} y^i, \quad F_i^{\,k} = 0, \\ B_{ij} = -\dfrac{1}{2\alpha}\left(a_{ij} - \dfrac{1}{\alpha^2} y_i y_j + 24 b_i b_j\right), \\ A^m_j = B^m_j b_{00} + B^m b_{j0}, \quad B^m_j = g^{mi} B_{ij} \\ \lambda^i = B^i b_{00}. \end{cases}$$

From (4.1), we have $B_{i0} = 0$ and $B^i_0 = 0$ which gives $A^m_0 = B^m b_{00.}$. Contracting (4.26) with y^k, we have

(4.28) $$D^i_{j0} = B^i b_{j0} + B^i_j b_{00} - B^m C^i_{jm} b_{00}.$$

Contracting (4.28) with y^j, we obtain

(4.29) $$D^i_{00} = B^i b_{00} = \left(\frac{5}{(1+4b^2)} b^i - \frac{1}{\alpha(1+4b^2)} y^i \right) b_{00}.$$

Contracting (4.28) by b_i and then using (4.1) & (4.27), we find

(4.30) $$b_i D^i_{j0} = \frac{5b^2}{(1+4b^2)} b_{j0} - \frac{(1+24b^2)}{2\alpha(1+4b^2)} b_{00} b_j - \frac{5}{(1+4b^2)} b^m C^i_{jm} b_i b_{00}.$$

Contracting (4.29) by b_i and using (4.1), we obtain

(4.31) $$b_i D^i_{00} = \frac{5b^2}{(1+4b^2)} b_{00}.$$

The covariant derivatives of the vector b_i with respect to x^j relative to the Cartan connection $C\Gamma$ and the Riemannian connection are given by

(4.32) (a) $b_{i|j} = \partial_j b_i - b_r F^r_{ij}$, (b) $b_{ij} = \nabla_j b_i = \partial_j b_i - b_r \left\{ {r \atop ij} \right\}$

respectively. Subtracting (4.32 a) and (4.32 b), we get

(4.33) $$b_{i|j} = b_{ij} - b_r D^r_{ij},$$

where $D^r_{ij} = F^r_{ij} - \left\{ {r \atop ij} \right\}$.

Contracting (4.33) by $y^i y^j$ and using (4.31), we obtain

(4.34) $$b_{i|j} y^i y^j = \left(\frac{1-b^2}{1+4b^2} \right) b_{00}.$$

Therefore, with the help of (4.34), (4.23) may be written as

(4.35) $$\left(\frac{b}{\sqrt{1+4b^2}} \right) H_0 + \left(\frac{1-b^2}{1+4b^2} \right) b_{00} = 0.$$

From (4.35), it is clear that the hypersurface $F_{n-1}(c)$ of the space F_n is hyperplane of first kind if and only if $b_{00} (= b_{ij} y^i y^j)$ vanishes. Since y^i satisfying (4.1) and quantity b_{ij} does not depend on the directional argument y^i, hence $b_{ij} y^i y^j = 0$ may be written as

(4.36) $$b_{ij} y^i y^j = (b_i y^i)(c_j y^j)$$

for some $c_j(x)$. Thus, we have

(4.37) $$2b_{ij} = b_i c_j + b_j c_i.$$

Transvecting (4.37) with $B^i_\alpha B^j_\beta$ and using (4.1), we get

(4.38) $$b_{ij}B^i_\alpha B^j_\beta = 0.$$

Again transvecting (4.37) with $B^i_\alpha y^j$ and using (4.1), we obtain

(4.39) $$b_{ij}B^i_\alpha y^j = 0.$$

From (4.27) and (4.37), we have

(4.40) $$\begin{cases} B_{ij}B^i_\alpha B^j_\beta = -\dfrac{1}{2\alpha}h_{\alpha\beta}, \ \ \lambda^i = 0, \\ A^i_j B^j_\beta = 0, \quad b_{i0}b^i = \dfrac{1}{2}b^2 c_0, \end{cases}$$

where $b_{i0} = b_{ij}y^j$.

From (3.13), (4.5), (4.7), (4.15), (4.26), (4.27), (4.38), (4.39) and (4.40), we have

(4.41) $$b_s D^s_{ij} B^i_\alpha B^j_\beta = -\frac{c_0 b^2 (b^2 - 1)}{4\alpha(1+4b^2)} h_{\alpha\beta}.$$

Since $b_{i|j}B^i_\alpha B^j_\beta = -b_s D^s_{ij}B^i_\alpha B^j_\beta$ for the hypersurface $F_{n-1}(c)$, from (4.21) and (4.41), we obtain

(4.42) $$\left(\frac{b}{\sqrt{1+4b^2}}\right) H_{\alpha\beta} + \frac{c_0 b^2 (b^2-1)}{4\alpha(1+4b^2)} h_{\alpha\beta} = 0.$$

This implies that $H_{\alpha\beta} \propto h_{\alpha\beta}$. Hence, we have

Theorem 4.3: *Let F_n be a Finsler space equipped with a metric (2.1). A hypersurface $F_{n-1}(c)$ of the Finsler space F_n is hyperplane of the first kind if and only if the condition (4.37) holds. Further in this case, the second fundamental $h-$tensor $H_{\alpha\beta}$ of the hypersurface $F_{n-1}(c)$ is proportional to its angular metric tensor $h_{\alpha\beta}$.*

According to Lemma 3.2, the necessary and sufficient condition for a hypersurface $F_{n-1}(c)$ of the Finsler space F_n is $H_{\alpha\beta} = 0$, i.e. the second fundamental $h-$tensor $H_{\alpha\beta}$ vanishes. Thus, from (4.42), we find

(4.43) $c_0 = c_i(x) y^i = 0$.

This implies that there exists a function $\varphi(x)$ which satisfies $c_i(x) = \varphi(x) b_i(x)$. Hence, equation (4.37) gives us

(4.44) $b_{ij} = \varphi(x) b_i b_j$.

Thus, we have

Theorem 4.4. *Let F_n be a Finsler space equipped with a metric (2.1). A hypersurface $F_{n-1}(c)$ of the Finsler space F_n is hyperplane of the second kind if and only if the condition (4.44) holds.*

Further, in view of Lemma 3.3 and (4.15) & (4.16), we find that the hypersurface $F_{n-1}(c)$ is not a hyperplane of the third kind.

Thus, we have

Theorem 4.5 *Let F_n be a Finsler space equipped with a metric (2.1).Then the hypersurface $F_{n-1}(c)$ of the Finsler space F^n is not a hyperplane of the third kind.*

References

1. Makoto Matsumoto, On C-reducible Finsler Spaces, *Tensor N. S.*, **24** (1972) 29-37.
2. Makoto Matsumoto, The Induced and Intrinsic Connections of a Hypersurface and Finslerian Projective Geometry, *J. Math. Kyoto Univ.*, **25** (1974) 477-498.
3. M. Hashiguchi and Y. Ichijyo, On Some Special $(\alpha, \beta)-$metric, *Reports of a Faculty of Science of Kagashima University*, **8** (1975) 39-46.
4. C. Shibata, On Finsler Spaces with an $(\alpha, \beta)-$metric, *Journal of HokkaidoUniversity of Education. Section II*, **35(1)** (1984) 1-16.
5. Makoto Matsumoto, Theory of Finsler spaces with $(\alpha, \beta)-$ metrics, *Rep. Math. Phys.* **31** (1992) 43-83.
6. H. Wogoughi, On a Hypersurface of Special Finsler Space with an exponential $(\alpha, \beta)-$ metric, *International Journal of Contemporary Mathematical Sciences*, **6** (2011) 1969-1980.
7. L. Y. Lee, H. Y. Park and Y. D. Lee, On a Hypersurface of a Special Finsler Space with a Metric $\alpha + \beta^2 / \alpha$, *Korean Journal of Mathematical Sciences*, **8(1)** (2001) 93-101.

8. M. K. Gupta and P. N. Pandey, On Hypersurface of a Finsler Space with Randers Conformal Metric, *Tensor N. S.*, **70(3)** (2008) 229-240.

9. M. K. Gupta, Abhay Singh and P. N. Pandey, On a Hypersurface of a Finsler Space with Randers Change of Matsumoto Metric, *Geometry*, **2013** (2013) 6pages.

10. Makoto Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaisheisha Press, Shigaken, 1986.

11. H. Rund, *The Differential Geometry of Finsler Spaces*, Springer-Verlag, 1959.