ISSN 0974 - 9373

Vol. 21 No. 2 (2017) **Journal of International Academy of Physical Sciences** pp. 123 -129

A Note on Fractional Order n-Species Lotka Volterra Competitive System with Delay

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(Received May 07, 2017)

Abstract: In this paper we discuss the fractional order n-species Lotka Volterra competitive species model with delay. Using the krasnoselskii's fixed point theorem we establish existence and uniqueness of a solution of the model under consideration.

Keywords: Fractional differential equation; Krasnoselskii's fixed point.

2000 Mathematics Subject Classification: 34K40, 34K14.

1. Introduction

It's a famous belief that the concept of fractional calculus has arisen from a question that was raised in the year 1965 by Marquis de L'Hospital (1661-1704) to Gottfried Wilhelm Leibniz (1646-1716), which sought the meaning of Leibniz's notation $\frac{d^n y}{dx^n}$ (n^{th} derivative) for the derivative of order $n \in \mathbb{N}_0 := \{0,1,2,.....\}$ when $n = \frac{1}{2}$ (What if $n = \frac{1}{2}$). In his reply, dated September 30, 1965, Leibniz wrote to L'Hospital as follows: "…This is an apparent paradox from which, one day, useful consequences will be drawn. …" In these words fractional calculus came into existence.

Further, fractional derivatives was mentioned in some contexts, like, for instance it was mentioned by Euler in 1730, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Liouville in 1832, Riemann in 1847, Greer in 1859, Holmgren in 1865, Grunwald in 1867, Letnikov in 1868, Sonin in 1869, Laurent in 1884, Nekrassow in 1888, Krug in 1890, and Weyl in

1917. In the two pages (pp. 409 – 410) by S. F. Lacroix on fractional calculus, the following was expressed,

$$\frac{d^{\frac{1}{2}}}{dv^{\frac{1}{2}}}v = \frac{2\sqrt{v}}{\sqrt{\pi}} .$$

Fractional Calculus is widely and efficiently used to describe many phenomena arising in engineering, physics, economy and science. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation[1]. Fractional differential equations, therefore find numerous applications in the field of visco-elasticity, feedback amplifiers, electrical circuits, electro analytical chemistry fractional muitipoles, neuron modeling encompassing different branches of physics, chemistry and biological sciences[2]. Many physical processes appear to exhibit fractional order behavior that may vary with time or space. The fractional calculus has allowed the operations of integration and differentiation to any fractional order. The order may take on any real or imaginary value. Recently, much attention has been paid to the existence of solutions for fractional differential equations, see[3, 4]. Most of them obtained results for solutions by using Schauder fixed-point theorem and banach contraction mapping principle. About the development of existence theorems for fractional functional differential equations, as far as we know, not much contribution exists except[1, 5, 6, 7, 8, 9]. Many applications of fractional calculus amount to replacing the time derivative in a given evolution equation by a derivatives seem to arise generally and universally from important mathematical reasons. Recently, an interesting attempt to give the physical meaning to the initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives was proposed in[10].

E.Ahmed et.al.[11] Considered the fractional-order predator prey model and the fractional-order rabies model. They have shown the existence and uniqueness of solutions of the model system and also studied the stability of equilibrium points. The motivation behind fractional order system is discussed in[11].

The purpose of this paper is to deal with the existence, uniqueness of solution of the following fractional order n-species Lotka Volterra system for $0<\alpha<1$,

(1.1) $$\frac{d^{\alpha}x_i(t)}{dt^{a}} = x_i(t)\left(r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t-\tau_{ij}(t))\right), \quad i=1,2,.......,n.$$

where d^{α} denotes Riemann-Liouville derivate of order α, $0<\alpha<1$ and $0\leq t\leq T$. We use Krasnoselskii's fixed point theorem to show the existence of a solution. We also use the contraction mapping principle.

2. Preliminaries and Results

The fractional integral of order $\alpha>0$ of a functional $f:R^{+}\to R$ of order $\alpha\in R^{+}$ is defined by

$$I_0^{\alpha} f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds,$$

provided the right side exists point wise on R^{+}. Γ is the gamma function. For instance, $I^{\alpha} f$ exists for all $\alpha>0$, when $f\in C^0(R^{+})\cap L^1_{loc}(R^{+})$; note also that when $f\in C^0(R_0^{+})$ then $I^{\alpha} f\in C^0(R_0^{+})$ and moreover $I^{\alpha} f(0)=0$.

The fractional derivative of order $\alpha>0$ of a functional $f:R^{+}\to R$ is given by

$$\frac{d^{\alpha}}{dt^{\alpha}} f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha} f(s)ds=\frac{d}{dt}I_0^{1-\alpha} h(t)\ .$$

Let X be a complex Banach space endowed the norm $\|.\|_X$. Denote $B(X)$ be the set of all bounded functions from R to X. We know that $B(X)$ is a Banach space endowed with the supremum norm denoted by $\|.\|_{B(X)}$.

Theorem 2.1: *(Krasnoselskii). Let Ω be a nonempty closed convex subset of a Banach space $\left(X,\|.\|\right)$. Suppose that F_1 and F_2 map Ω into X such that*

(i) For any $x, y\in\Omega$, $F_1 x + F_2 y\in M$,

(ii) F_1 is a contraction,

(iii) F_2 is continuous and $F_2(M\Omega)$ is contained in a compact set.

Then there exists $z\in M\Omega$ such that $z=F_1 z+F_2 z$.

Consider *n*-species fractional order competitive species model

$$\frac{d^{\alpha}x_i(t)}{dt^{\alpha}}=x_i(t)\Big(r_i(t)-\sum_{j=1}^{n}a_{ij}(t)x_j\big(t-\tau_{ij}(t)\big)\Big),\quad i=1,2,\ldots\ldots.,n. \tag{2.1}$$

Using fractional calculus above equation can be represent as an integral form

$$x_i(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}x_i(s)\Big(r_i(s)-\sum_{j=1}^{n}a_{ij}(s)x_j\big(s-\tau_{ij}(s)\big)\Big)ds,\quad i=1,2,\ldots\ldots.,n. \tag{2.2}$$

Define a mapping F_i given by

$$(Fx)_i(t)=(Fx)_{i1}(t)+(Fx)_{i2}(t)\,,$$

where

$$(Fx)_{i1}(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}x_i(s)r_i(s)\,ds$$

and

$$(Fx)_{i2}(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}x_i(s)\sum_{j=1}^{n}a_{ij}(s)x_j\big(s-\tau_{ij}(s)\big)ds.$$

Define

$$X=\big\{x(t)=\big(x_1(t),x_2(t),\ldots\ldots..,x_n(t)\big)\in C(R,R^n)\big\}$$

and

$$\|x\|=\sum_{i=1}^{n}\sup_{t\in R}|x_i(t)|,\qquad i=1,2,\ldots\ldots.,n.$$

Thus X is a Banach space endowed with the above norm.
Now we show the following result.

Theorem 2.2: *The system (1.1) has at least one solution for* $\|r\|\le 1$.

Proof: In order to use Krasnoselskii's fixed point theorem, one need to construct two mapping, a contraction map and a map which satisfy the second condition. For $x,\,y\in X$ we have

$$\|(Fx)_{i1}(t)-(Fy)_{i1}(t)\|_X\le\frac{1}{\Gamma(\alpha)}\int_0^t|(t-s)|^{\alpha-1}|x_i(s)-y_i(s)||r_i(s)|\,ds \tag{2.3}$$

$$\le\frac{1}{\Gamma(\alpha)}\|x_i-y_i\|\,\|r_i\|\int_0^t t^{\alpha-1}\,ds.$$

Thus for $\|r\|=\sum_{i=1}^{n}\sup_{t\in R}|r_i(t)|<1$, we have

$$\|F_1x-F_1y\| \leq \|x-y\|\,\|r\|<\|x-y\|$$

implies that F_1 is a contraction.

Let us define the set $Q=\left\{x\in R^n : \|x_i\|\leq M+\frac{1}{\Gamma(\alpha)}M^2\alpha\right\}$. One can easily see that the set Q is closed and convex. Now consider

$$\begin{aligned}(2.4)\qquad \|(F_ix)(t)\| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t|(t-s)|^{\alpha-1}\left(|x_i(s)\|r_i(s)|+|x_i(s)|\sum\nolimits_{j=1}^n|a_{ij}(s)|\left|x_j\left(s-\tau_{ij}(s)\right)\right|\right)ds\\ &\leq \frac{1}{\Gamma(\alpha)}\|x\|\,\|r\|\int_0^t s^{\alpha-1}ds+M^2\sum\nolimits_{j=1}^n\int_0^t|a_{ij}(s)|ds\\ &\leq M\|r\|+\frac{1}{\Gamma(\alpha)}M^2\alpha\\ &\leq M+\frac{1}{\Gamma(\alpha)}M^2\alpha.\end{aligned}$$

For $x\in Q$ and $0<t_2<t_1$, we have

$$\begin{aligned}(2.5)\qquad \|(Fx)_{i2}(t_1)-(Fx)_{i2}(t_2)\| &= \left\|\begin{array}{l}\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}x_i(s)\sum\nolimits_{j=1}^n a_{ij}(s)\,x_j\left(s-\tau_{ij}(s)\right)ds\\ -\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}x_i(s)\sum\nolimits_{j=1}^n a_{ij}(s)\,x_j\left(s-\tau_{ij}(s)\right)ds\end{array}\right\|\\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_2}\left((t_2-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right)\|x_i(s)\|\sum\nolimits_{j=1}^n|a_{ij}(s)|\,\left\|x_j\left(s-\tau_{ij}(s)\right)\right\|ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_2}^{t_1}\left((t_1-s)^{\alpha-1}\right)\|x_i(s)\|\sum\nolimits_{j=1}^n|a_{ij}(s)|\,\left\|x_j\left(s-\tau_{ij}(s)\right)\right\|ds\\ &\leq \frac{1}{\alpha}M^2\|\alpha\|\left|t_1^\alpha-t_2^\alpha-2(t_1-t_2)^\alpha\right|\\ &\leq \frac{2}{\alpha}M^2\|\alpha\|(t_1-t_2)^\alpha.\end{aligned}$$

Thus $F_1u+F_2v\in Q$. Hence all the conditions of Kranoselkii's theorem are satisfied. Thus there exists a fixed point $\Phi\in Q$ such that $\Phi=\Lambda_1\Phi+\Lambda_2\Phi$. Hence this is the solution of (1.1).

3. Example

Consider the following two species of interacting populations without inter specific competition. Thus consider the following parametric value:

$$r_1(t)=\frac{1}{2},\ r_2(t)=\frac{1}{3},\ a_{11}(t)=0,\ a_{22}(t)=0,\ a_{12}(t)=\sin(t), a_{21}(t)=\cos(t).$$

We have the following model system for $\alpha=\frac{1}{2}$ and $\tau=\tau_{ij}=0.1,$ for $i, j=1, 2$

$$\frac{d^{1/2}x_1(t)}{dt^{1/2}}=x_1(t)\left(\frac{1}{2}-a_{12}(t)x_2(t-0.1)\right),$$

$$\frac{d^{1/2}x_2(t)}{dt^{1/2}}=x_2(t)\left(\frac{1}{2}-a_{21}(t)x_1(t-0.1)\right).$$

One can easily see that $|r|=\frac{1}{2}+\frac{1}{3}=\frac{5}{6}<1$ and $\alpha=\left|\int_0^t(\sin(s)+\cos(s))ds\right|<2$.

Thus for a particular choice of M our model system has a unique solution in the set

$$Q=\left\{x=(x_1, x_2):\|x\|\le M+\frac{2}{\sqrt{\pi}}M^2\right\}$$

according to Theorem 2.2.

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