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Stability Conditions for Fixed Point Iteration Procedures

Neeta Singh
Department of Mathematics
University of Allahabad, Allahabad-211002, India
Email: n_s32132@yahoo.com

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Abstract: In this paper, we study conditions under which certain fixed-point iteration procedures are stable. We also discuss the rate of convergence of the iterations and calculate the value of the iteration parameter for which the convergence rate is fastest.

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1. Introduction

The aim of this work is to study the conditions under which certain fixed-point iteration procedures are stable. We will first recall all the fixed-point iteration procedures whose stability we will study. Intuitively, a fixed point iteration procedure is called numerically stable if for any approximate sequence $\{y_n\}$ for $\{x_n\}$, i.e. y_n is in some sense close enough to x_n at each stage and $\{y_n\}$ still converges to the fixed point of T, where the fixed point iteration procedure is given by a general relation of the form

$$x_{n+1} = f(T, x_n), \quad n = 0, 1, 2 \ldots\ldots\ldots\ldots$$

where $T : X \to X$ is an operator and $x_0 \in X$ and (X, d) is a metric space with $F(T) \neq \phi$ where $F(T)$ is the set of fixed points of the operator T and $\{x_n\}_{n=0}^{\infty}$ is a sequence obtained by a certain fixed point iteration procedure which converges to a fixed point of T. Let us discuss the various fixed-point iteration procedures we will be using in this work.

Let (X, d) be a metric space, $D \subset X$ a closed subset of X and $T : D \to D$ has at least one fixed point, say p, i.e. $p \in F(T)$ where $F(T)$ denotes the fixed point set of the operator T. For a given $x_0 \in X$, we consider the sequence of iterates $\{x_n\}_{n=0}^{\infty}$ determined by the successive of iteration method

$$x_n = \mathrm{T}(x_{n-1}) = T^n(x_0), \quad n = 1, 2 \ldots\ldots\ldots\ldots$$

The main interest lies in obtaining conditions on T, D and X, which can generate iterates $\{x_n\}_{n=0}^{\infty}$ that converge to a fixed point of T in D. In most cases $D = X$. Generally by convergence we will mean strong convergence but the concept of weak convergence is also being considered where strong convergence cannot be obtained.

The sequence $x_n = T(x_{n-1}), \; n = 1, 2, \ldots\ldots\ldots\ldots\ldots$ is called the Picard iteration. In many situations the Picard iteration does not converge or even if it converges, it does not converge to a fixed point of the operator T, then some other iteration procedures are considered. Let E be a real normed space and $T : E \to E$ be a self-map, $x_0 \in E$ and $\lambda \in [0,1]$. The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\lambda) x_n + \lambda T x_n, \; n = 0, 1, 2 \ldots\ldots\ldots$$

is called the Krasnoselskij iteration or the $K-$iteration. For $\lambda = 1$, this iteration reduces to the Picard iteration.

The Mann iteration is defined as follows. For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-a_n)\, x_n + a_n T x_n, \; n = 0, 1, 2 \ldots\ldots\ldots\ldots\ldots$$

where $\{a_n\}_{n=0}^{\infty} \subset [0,1]$ satisfies certain conditions is called the Mann iteration. If $a_n = \lambda$ (constant) the Mann iteration reduces to the Krasnoselskij iteration. The Ishikawa iteration is described as follows. Let $x_0 \in X$ and

$$x_{n+1} = (1-a_n)\, x_n + a_n T \left[(1-b_n) x_n + b_n T x_n \right], \; n = 0, 1, 2, \ldots\ldots\ldots\ldots\ldots$$

where $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty} \subset [0,1]$ satisfy certain conditions.

The Ishikawa iteration was first applied to obtain strong convergence to a fixed point of T for a Lipschitzian and pseudo-contractive operator of a convex, compact subset of a Hilbert space. In this situation the Mann iteration failed to converge. We write this iteration as a type of two-step Mann iteration.

$$x_{n+1} = (1-a_n)x_n + a_n T y_n, \quad n = 0,1,2,...............$$

$$y_n = (1-b_n)x_n + b_n T y_n.$$

For $b_n = 0$, the Ishikawa iteration reduces to the Mann iteration. Another iteration procedure which has led to important results in the Krasnoselski-Mann-Opial (K-M-O) iteration procedure which is defined as follows:
For any given $x_0 \in X$, the iteration sequence $\{x_n\}$ is defined by

$$x_{n+1} = (1-\alpha)x_n + \alpha\, S x_n, \quad n \geq 0$$

where the iteration parameter $\alpha \in (0,1)$ and $S : D \subseteq X \to X$ is an operator.

Let us now denote a fixed-point iteration procedure by

$$x_{n+1} = f(T, x_n), \quad n = 0,1,2............$$

where $T : X \to X$ is an operator and $x_0 \in X$. $f(T, x_n)$ contains all the parameters that define the given fixed point iteration procedure.

Definition[2] 1.1: Let (X, d) be a metric space and $T : X \to X$ be a mapping. Let $x_0 \in X$ and we assume that the iterates given by

$$x_{n+1} = f(T, x_n), \quad n = 0,1,2............$$

converges to a fixed point p of T. Let $\{y_n\}$ be an arbitrary sequence in X and define

$$\varepsilon_n = d(y_{n+1}, f(T, y_n)) \quad \text{for} \quad n = 0,1,2,........$$

We call the above iteration procedure T-stable or stable with respect to T if and only if $\lim_{n\to\infty} \varepsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} y_n = p.$

In this work we will also discuss the rate of convergence of the iteration i.e. for which value of the iteration parameter $\alpha(0<\alpha<1)$, in the K-iteration, the convergence rate is fastest.

2. Stability Conditions for Krasnoselskij Iteration

First we recall the KMO iteration. Let $x_0 \in X, \{x_n\}$, a sequence in X, where X is a real Banach space

$$x_{n+1}=(1-\alpha)x_n+\alpha S x_n,\ n\ \geq\ 0\ \ \alpha\in(0,1)$$

This result is an improvement of Theorem 8.3.2 (a) since we drop the condition $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)=\infty$. Let us recall the Theorem which is by Dotson Jr.[3]

Theorem 2.1: *Let H be a Hilbert space and $T:H\to H$ a monotonic nonexpansive operator. For $f\in H$, define $S:H\to H$ by $S(x)=-Tx+f$, $x\in H$. Then the Mann iteration defined by $x_{n+1}=M(x_n,\alpha_n,S)$ with $\alpha_n\in[0,1]$, $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)=\infty$, converges strongly to the unique solution $x=v$ of the operator equation $x+Tx=f$.*

From $Sx=-Tx+f$ we have $\|Sx-Sy\|=\|Tx-Ty\|\leq\|x-y\|$ for all $x,y\in H$. The solution v of the operator equation $x+Tx=f$ is a fixed point of the non-expansive operator S. By the monotonicity of T, we get

$$\langle Sx-Sy,\ x-y\rangle=-\langle Tx-Ty,x-y\rangle\leq 0 \quad \text{for all } x,y\in H.$$

Since $\alpha(1-\alpha)>0$, we have

$$\|x_{n+1}-v\|^2<[1-2\alpha(1-\alpha)]\|x_n-v\|^2<[1-2\alpha(1-\alpha)]^2\|x_{n+1}-v\|^2$$
$$<[1-2\alpha(1-\alpha)]^n\|x_1-v\|^2.$$

Now

$$[1-2\alpha(1-\alpha)]^n\leq\exp[-2n\alpha(1-\alpha)]=\frac{1}{e^{2n\alpha(1-\alpha)}}\to 0 \text{ as } n\to\infty.$$

Hence $\{x_n\}$ converges strongly to the unique solution of the operator equation $x+Tx=f$. In the case of this iteration

$$\phi(\alpha)=1-2\alpha(1-\alpha)\ =1-2\alpha+2\alpha^2$$

giving

$$\frac{d\phi}{d\alpha}=-2+4\alpha=0\ \Rightarrow\alpha=\frac{1}{2}.$$

Also, $$\frac{d^2\phi}{d\alpha^2}=4>0.$$

Hence minima is attained for $\alpha = 1/2$. This iteration converges fastest for $\alpha = 1/2$ in the family $(0 < \alpha < 1)$. Next, we examine the KMO iteration for stability.

Since the operator is non-expansive, the Lipschitz constant $L = 1$ now putting $L = 1$ in Prop[5] we get,

(2.1) $$\|z_{n+1} - q\| \le \|z_{n+1} - p_n\| + \|p_n - q\| = \varepsilon_n + \frac{17}{18}\|z_n - q\|$$

and

(2.2) $$\varepsilon_n = \|z_{n+1} - p_n\| \le \|z_{n+1} - q\| + \|p_n - q\| \le \|z_{n+1} - q\| + \frac{17}{18}\|z_n - q\| .$$

Lemma[2] 2.2: *Let* $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ *be sequences of nonnegative numbers and* $0 \le q < 1$, *so that*

$$a_{n+1} \le q a_n + b_n \quad \textit{for all } n \ge 0$$

Then if $\lim_{n\to\infty} b_n = 0$ *then* $\lim_{n\to\infty} a_n = 0$. *Now* $\lim_{n\to\infty}\|z_n - q\| = 0 \Rightarrow \lim_{n\to\infty}\|z_{n+1} - q\| = 0.$

Applying (2.2) gives $\lim_{n\to\infty}\varepsilon_n \le 0 \Rightarrow \lim_{n\to\infty}\varepsilon_n = 0$. Conversely put $\|z_{n+1} - q\| = a_{n+1}$ and $\|z_n - q\| = a_n$. Applying (2.1) we get

$$a_{n+1} \le \varepsilon_n + \frac{17}{18}a_n \text{ or } a_{n+1} \le \frac{17}{18}a_n + \varepsilon_n .$$

Put $b_n = \varepsilon_n$ and $q = \frac{17}{18}$. Since $0 \le \frac{17}{18} < 1$ we get $\lim_{n\to\infty}\varepsilon_n = 0$ giving $\lim_{n\to\infty} z_n = q$. Thus $\lim_{n\to\infty} z_n = q \Leftrightarrow \lim_{n\to\infty}\varepsilon_n = 0.$

Hence the K-M-O type iteration is S-stable. The second result of this section is that we test the stability of the K-iteration procedure given in following theorem.

Theorem[2] 2.3: *Let* C *be a bounded closed convex subset of a Hilbert space* H *and* $T : C \to C$ *be a non-expansive and demicompact operator. Then the set* $F(T)$ *of fixed points of* T *is a nonempty convex set and for any given* x_0 *in* C *and any fixed number* λ *with* $0 < \lambda < 1$, *the K- iteration* $\{x_n\}$ *given by*

$$x_{n+1}=(1-\lambda)x_n+\lambda T x_n,\ n=0,1,2,........$$

converges strongly to a fixed point of T.

Now since the mapping is non-expansive, the Lipschitz constant $L=1$. From[5] we get that for non-expansive mappings.

$$\|x_{n+1}-q\| \le \left(\frac{17}{18}\right)^{n+1}\|x_0-q\| \qquad \text{for all } n\ge 0.$$

q is a fixed point of operator S defined by

$$S(x)=f-Ax,\ f\in H.$$

Let $\{z_n\}$ be any sequence in H Putting $p_n=(1-\alpha)z_n+\alpha S z_n \quad (n\ge 0)$

we have by[5] that

(2.3) $$\|z_{n+1}-q\|\le\varepsilon_n+\frac{17}{18}\|z_n-q\|$$

and

(2.4) $$\varepsilon_n\le\|z_{n+1}-q\|+\frac{17}{18}\|z_n-q\|.$$

Let us assume that $\lim_{n\to\infty}\|z_n-q\|=0.$ By inequality (2.4) we get that $\lim_{n\to\infty}\varepsilon_n\le 0\Rightarrow\lim_{n\to\infty}\varepsilon_n=0.$ Conversely let us assume that $\lim_{n\to\infty}\varepsilon_n=0.$ We recall the following Lemma

Lemma[2] 2.4: *Let* $\{a_n\}_{n=0}^{\infty},\{b_n\}_{n=0}^{\infty}$ *be sequences of nonnegative numbers and* $0\le q<1$ *so that* $a_{n+1}\le q a_n+b_n$ *for all* $n\ge 0$. *If* $\lim_{n\to\infty}b_n=0,$ *then* $\lim_{n\to\infty}a_n=0.$

Let us apply this Lemma to inequality (2.3), since $0\le\frac{17}{18}<1$ and $\lim_{n\to\infty}\varepsilon_n=0,$ we conclude that $\lim_{n\to\infty}\|z_n-q\|=0.$ Hence $\lim_{n\to\infty}z_n=0\Leftrightarrow\lim_{n\to\infty}\varepsilon_n=0.$ By definition of stability the K - M - O type iteration is S - stable. From[5] we get that $Q(\alpha)=\frac{1+(L+1)^2\alpha^2}{1+\alpha}.$ Putting $L=1, Q(\alpha)=\frac{1+4\alpha^2}{1+\alpha}.$ The convergence rate is fastest for $\alpha = \frac{1}{2(L+1)^2}$. Putting $L=1$, we get $\alpha = \frac{1}{8}$. Hence for $\alpha = \frac{1}{8}$, the K-iteration converges fastest in the family $0<\alpha<1$.

Same arguments as above will apply to the following theorems dealing with the K-iterations and concluding that the K-iterations in these cases will be stable. In all the following theorems, the mapping is non-expansive and therefore the Lipschitz constant $L=1$.

Theorem 2.5: *Let X be a uniformly convex Banach space D a closed bounded convex set in X and T a non-expansive mapping of D into D such that T satisfies any one of the following two conditions:*

(i) (I-T) maps closed sets in D into closed sets in X ;

(ii) T is demicompact at 0.

Then for any given x_0 in C and any fixed number λ with $0<\lambda<1$, the K-iteration $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 2.6: *Suppose E is a real Banach space and $F: E \to E$ is a Lipschitzian strongly accretive operator. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences satisfying*

$$(i) \quad 0 \le \alpha_n \beta_n < 1, \; n \ge 0 \; (ii) \quad \lim_{n\to\infty} \alpha_n = 0, \lim_{n\to\infty} \beta_n = 0 \; (iii) \; \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ generated starting from any $x_0 \in E$ by

$$y_n = (1-\beta_n)x_n + \beta_n\left(f + (I-F)x_n\right), \qquad n \ge 0$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\left(f + (I-F)y_n\right), \qquad n \ge 0$$

converges strongly to the solution of the equation $Fx = f$.

Putting $\beta_n = 0$, we get $y_n = x_n$ and $\alpha_n = \alpha \quad \forall n$, we get the K-iteration as before. The same arguments for stability will apply to the following Theorem.

Theorem 2.7: *Suppose E is a real Banach space and $F: E \to E$ is a Lipschitzian accretive operator. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences* *satisfying (i)-(iii) as in previous theorem. Then the sequence* $\{x_n\}$ *generated from an arbitrary* $x_0 \in X$ *by*

$$y_n = (1-\beta_n)x_n + \beta_n(f - Tx_n), \qquad n \ge 0$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(f - Ty_n), \qquad n \ge 0$$

converges strongly to the unique solution of the equation $x + Fx = f, f \in E.$

Theorem 2.8: *Let* X *be a strictly convex Banach space,* C *a closed bounded convex subset of* X *and* $f : C \to C$ *a densifying nonexpansive mapping, let* $f_\lambda(x) = \lambda x(1-\lambda)f(x)$ *for constant* λ *with* $0 < \lambda < 1,$ *then for each* $x_n \in C,$ *the sequence*

$$x_{n+1} = \lambda x_n + (1-\lambda) f x_n, \; n = 0,1,2,.......$$

converges strongly to a fixed point of f *in* $C.$

3. Stability Conditions for Mann Iteration

This section deals with the stability of the Mann iteration procedure. Let us recall the following definition and theorem. Let (X, d) be a metric space and $T : X \to X$ satisfies the condition

(3.1) $$d(Tx, Ty) \le a\, d(x, y) + L\, d(x, Tx)$$

for some $a \in [0,1], L \ge 0$ when L is the Lipschitz constant, for all $x, y \in D \subset X.$

Definition[2] 3.1*:* In a normed space $E,$ on operator $T : E \to E$ is called Zamfirescu operator if there exist numbers α, β and $\gamma, 0 \le \alpha < 1, 0 \le \beta, \gamma < 0.5$ such that for any $x, y \in E$ at least one of the following conditions is true :

(z1) $\|Tx - Ty\| \le \alpha \|x - y\|;$

(z2) $\|Tx - Ty\| \le \beta \left[\|x - Tx\| + \|y - Ty\| \right];$

(z3) $\|Tx - Ty\| \le \gamma \left[\|x - Tx\| + \|y - Ty\| \right].$

Theorem 3.2: *Let* E *be a normed linear space and* $T:E\to E$ *a mapping satisfying condition (3.1). Suppose* T *has a fixed point* x^*. *Let* x_0 *by arbitrary in* E *and define*

$$z_n=(1-\beta_n)x_n+\beta_n T x_n, \qquad n\geq 0$$

and

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n T z_n, \qquad n\geq 0$$

where $\{\alpha_n\},\{\beta_n\}$ *are sequences in* $[0,1]$ *such that* $0<\alpha\leq\alpha_{n+1}$ *for some* α. *Let* $\{y_n\}$ *be any given sequence in* E *and define*

$$s_n=(1-\beta_n)y_n+\beta_n T y_n, \quad n\geq 0$$

$$\varepsilon_n=\left\|y_{n+1}-(1-\alpha_n)y_n-\alpha_n T s_n\right\|, \quad n\geq 0 .$$

Then $\{x_n\}$ *converges strongly to* x^* *and is stable with respect to* T.

Now let us recall the following theorem.

Theorem[2] 3.3: *Let* E *be a uniformly convex Banach space,* K *a closed convex subset of* E *and* $T:K\to K$ *be a Zamfirescu mapping. Then the Mann iteration* $\{x_n\}$,

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n T x_n, \qquad n=1,2,......$$

with $\{\alpha_n\}$ *satisfying the conditions:*

(i) $\alpha_1=1$

(ii) $0\leq\alpha_n<1$ *for* $n>1$

(iii) $\sum\alpha_n(1-\alpha_n)=\infty$

converges to the unique fixed point of T.

We now show that $\{x_n\}$ is stable with respect to T by applyingTheorem[2]. Putting $L=0$ in condition (2.3) we set

$$d(Tx,Ty)\leq a\,d(x,y) \text{ where } 0\leq a<1$$

which is the first condition (z_1) for Zamfirescu mapping. We take $\beta = 0$ and $\gamma = 0$ giving $d(Tx, Ty) \le 0$. Hence $\{x_n\}$ is stable with respect to T. Using the same arguments as above we can prove the stability of the Mann iteration in the Theorem below.

Theorem[2] 3.4: *Let E be an arbitrary Banach space, K a closed convex subset of E and $T : K \to K$ an operator satisfying conditions $(z_1), (z_2)$ and (z_3) with $d(x, y) = \|x - y\|$. Let $\{x_n\}$ be a Mann iteration with $\{a_n\} \subset [0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Same arguments as above can be used to prove stability of the iteration in the following Theorem.

Theorem 3.5: *Let K be a nonempty closed convex subset of a banach space E and $T : K \to K$ a quasicontration, suppose $\alpha_n > 0$ for all $n \ge 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Let $\{x_n\}$ be the sequence defined by $x_0 \in K$,*

$$y_n \in Co\{x_i\}_{i=k_n}^{n} \cup \{Tx_i\}_{i=k_n}^{n}, \qquad n \ge 0$$

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T y_n, \qquad n \ge 0,$$

where $\{k_n\}$ is a non-decreasing sequence of positive integers such that $k_n \le n$ and $\lim_{n\to\infty} k_n = \infty$. Then $\{x_n\}$ converges strongly to the unique fixed point of T.

By[2], quasi - contractive operators contain zamfirescu operators.

Definition 3.6: A mapping in a normed space E is called quasi-contraction if

$$\|Tx - Ty\| \le kM(x, y), \quad x, y \in E, \text{ where}$$

$$M(x, y) = \max\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\}.$$

Since non-expansive mapping is a particular case of accretive mapping by applying[5] we get the stability of the Mann iteration in the following results.

Theorem[1] 3.7: *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and T a non-expansive mapping of C into a compact subset of C. Let $x_1 \in C$ be an arbitrary point in C, then the sequence $\{x_n\}$ defined by*

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n) = M\left(x_n, \frac{1}{2}, T\right)$$

converges strongly to a fixed point T in C.

Theorem[1] 3.8: *Let C be a nonempty closed subset of a banach space X and let T be a non-expansive mapping from C into a compact subset of X. Suppose that there exist $x_1 \in C$ and a sequence $\{x_n\}$ of real numbers satisfying*

(i) $0 \le \alpha_n \le \alpha < 1$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$

(ii) $x_n \in C$ *for all* $n \in N$, *where* $x_{n+1} = M(x_n, \alpha_n, T)$,

then $\{x_n\}$ converges strongly to an element of $F(T)$.

Definition[2] 3.9: Suppose E is a real Banach space and T is a self-map of E with $F(T) \neq \phi$. Let $x_0 \in E$ and let $\{x_n\}$ be an iteration procedure given by

$$x_{n+1} = f(T, x_n), \quad n = 0, 1, 2, ...$$

that converges strongly to a fixed point $x^* \in F(T)$. Let $\{y_n\}$ be a sequence in E and $\{\varepsilon_n\}$ is a sequence of positive real numbers given by

$$\varepsilon_n = \left\| y_{n+1} - f(T, y_n) \right\|$$

If $\sum_{n=0}^{\infty} \varepsilon_n < \infty \Rightarrow \lim_{n \to \infty} y_n = x^*$ then the iteration procedure defined above is called almost $T-$stable.

Definition[2] 3.10: Let E be a Banach space, K a subset of E and $T : K \to K$ is called a strongly pseudocontractive operator if there exists a number $t > 1$ such that the inequality

$$\|x - y\| \le \|(1+r)(x-y) - rt(Tx - Ty)\| \text{ holds for all } x, y \in K \text{ and } r > 0.$$

We now show that the fixed-point iteration used in the following theorem is almost $T-$stable.

Theorem[2] 3.11: *Let E be a Banach space and K a nonempty closed convex and bounded subset of E. If $T:K\to K$ is a Lipschitzian strongly pseudocontractive operator such that the fixed point set of $T, F(T)$ is nonempty, then the Mann iteration $\{x_n\}\subset K,\ x_1\in K$ and the sequence $\{a_n\}\subset(0,1]$, with*

$$(i)\ \sum_{n=1}^{\infty}\alpha_n=\infty \quad \text{and} \quad (ii)\ \alpha_n\to 0\ (\text{as } n\to\infty)$$

converges strongly to the unique fixed point (say p) of T.

Now[2] gives that any strongly pseudocontractive operator is pseudo $\phi-$contraction for certain given rules of a,b,c and $\phi(u)$. Applying[2] we get that the Mann iteration (which is a special case of the Ishikawa iteration) is almost $T-$stable. i.e.

$$\sum_{n=0}^{\infty}\varepsilon_n<\infty\Rightarrow\lim_{n\to\infty}y_n=p.$$

Using the same arguments as above we can show that the iteration procedure in the following theorem is almost $T-$stable.

Theorem 3.12: *Let E be a real uniformly smooth Banach space and K a bounded closed convex and nonempty subset of E. Let $T:K\to K$ be a strongly pseudo contractive operator such that $T(p)=p$ for some $p\in K$ and let $\{x_n\}$ be the Mann iteration process generated by $x_1\in K$ and the sequence $\{a_n\}$ satisfying the following conditions:*

(i) $0\le\alpha_n<1$ *for all* $n\ge 1$;

(ii) $\lim_{n\to\infty}\alpha_n=0$;

(iii) $\sum_{n=0}^{\infty}\alpha_n=\infty.$

Then, for arbitrary $x_1\in K$, the sequence $\{x_n\}$ converges strongly to p and p is unique.

4. Stability Conditions for Ishikawa Iteration

In this section we will study the stability of the Ishikawa iteration process described in the Introduction. Let us recall the following theorem

Theorem[2] 4.1: *For E a normed linear space and $T:E\to E$ a mapping satisfying.*

$$d(Tx,Ty)\le a\,d(x,y)+L\,d(x,Tx) \text{ for some } a\in[0,1],\ \ L\ge 0,\ \ x,y\in D\subset X.$$

Let $x_{n+1}=(1-\alpha_n)x_n+\alpha_n T z_n,\quad n\ge 0$ *where* $z_n=(1-\beta_n)x_n+\beta_n T x_n,\quad n\ge 0$ *where* $\{\alpha_n\},\{\beta_n\}\subset[0,1]$. *Let* $\{y_n\}$ *be any given sequence in* E *and define*

$$S_n=(1-\beta_n)y_n+\beta_n T y_n,\quad n\ge 0$$

$$\varepsilon_n=\left\|y_{n+1}-(1-\alpha_n)y_n-\alpha T s_n\right\|,\quad n\ge 0$$

Then $\{x_n\}$ *converges strongly to* x^*, *a fixed point of* T *and is stable with respect to* T.

We apply this theorem to obtain stability of the Ishikawa iteration in the following result.

Theorem[2] 4.2: *Let E be a uniformly convex Banach space, K a closed convex subset of E and $T:K\to K$ a Zamfirescu operators. Let $\{\alpha_n\},\{\beta_n\}\subset[0,1]$ with $\{\alpha_n\}$ satisfying the condition $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)$ diverges. Then for any $x_0\in K$, the Ishikawa iteration process $I(x_0,\alpha_n,\beta_n,T)$ converges strongly to the unique fixed point of T.*

The condition $d(Tx,Ty)\le ad(x,y)+Ld(x,Tx)$ reduces to condition (z_1) for the Zamfirescu operator when $L=0$, and $0\le a<1$ Hence we conclude that the Ishikawa iteration is T-stable. We apply[2] to show almost stability of the Ishikawa iteration in the following results.

Theorem[1] 4.3: *Let C be a nonempty compact convex subject of a Hilbert space H and $T:C\to C$ a Lipschitzian Pseudo contractive mapping with $F(T)\ne\phi$. Let $\{x_n\}$ be an Ishikawa iteration, then $\{x_n\}$ converges strongly to a fixed point of T.*

Theorem[2] 4.4: *Let* K *be a convex compact subset of a Hilbert space* H *and* $T:K\to K$ *be a Lipschitzian pseudo contractive map and* $x_1\in K$. *Then the Ishikawa iteration* $\{x_n\}$, *where* $\{\alpha_n\},\{\beta_n\}$ *satisfy*

(i) $0\le\alpha_n\le\beta_n\le 1,\ n\ge 1$

(ii) $\lim_{n\to\infty}\beta_n=0$

(iii) $\sum_{n=1}^{\infty}\alpha_n\beta_n=\infty,$

converges strongly to a fixed point of T.

Theorem[2] 4.5: *Let* E *be a real uniformly smooth Banach space and* K *a bounded closed convex and nonempty subset of* E *Let* $T:K\to K$ *be a strongly pseudo contractive operator that has at least a fixed point* $x^*\in F(T)$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$ *satisfy*

(i) $0\le\alpha_n,\beta_n\le 1,\ \ n\ge 0$

(ii) $\lim_{n\to\infty}\alpha_n=0, \lim_{n\to\infty}\beta_n=0$

(iii) $\sum_{n=1}^{\infty}\alpha_n=\infty.$

Then for arbitrary $x_0\in K$, *the Ishikawa iteration converges strongly to* x^*, *also* x^* *is unique.*

5. Comparison of Convergence Rate of Various Iterations

In this section we compare the convergence rate of the different iteration procedures. Let us recall the following Proposition from[5].

Proposition[5] 5.1: *Let* X *be a real Banach space and* $A:X\to X$ *be a lipschitz accretive operator with Lipschitz constant* $L(L\ge 1)$, *then the sequence* $\{x_n\}$ *defined by*

$$x_{n+1}=(1-\alpha)x_n+\alpha Sx_n, \qquad n\ge 0,$$

$\alpha\in(0,1)$ *is an iterative solution of* $x+Ax=f,\ x,f\in X$ *x, f Є X where the iterative parameter* $\alpha_0\in\left(0,\dfrac{1}{(L+1)^2}\right)$.

In this case $Q(\alpha)=1+(L+1)^2\alpha^2/1+\alpha,$ where the above iteration is the K-M-0 iteration. Now for the Picard iteration $\alpha=1,$ i.e. $x_{n+1}=Sx_n$ and $Q(\alpha)=Q(1)=1+(L+1)^2/2$. The K-M-O iteration converges fastest for $\alpha_0=1/2(L+1)^2$. Putting this value of α_0 in $Q(\alpha)$ we get $Q(\alpha_0)=4L^2+8L+5/4L^2+8L+6$. Since $L\geq 1,$ $\frac{4L^2+8L+5}{4L^2+8L+6}<\frac{1+(L+1)^2}{2}.$ Hence in this case the K-M-O iteration converges faster than the Picard iteration.

Next, recall the following Theorem for Mann iteration.

Theorem[2] 5.2: *Let E be a arbitrary Banach space, K a closed convex subset of E and $T:K\to K$ an operator satisfying conditions $(z_1),(z_2)$ and (z_3). Let $\{x_n\}$ be the Mann iteration with $x_0\in K,$ $\alpha_n\subset[0,1]$ satisfying $\sum_{n=0}^{\infty}\alpha_n=\infty.$ Then $\{x_n\}$ converges strongly to the unique fixed point of T.*

In this case

$$Q(\alpha_n)=1-(1-\delta)\alpha_n$$

and the convergence of the Picard iteration can be obtained from the above theorem by putting $\alpha_n=1,$ because of the four restrictive assumptions, giving $Q(\alpha_n)=\delta.$ By $(z_1),(z_2),(z_3)$ and $a=\alpha,b=\beta,c=\gamma,$ we find

$$\delta=\max\left\{a,\frac{b}{1-b},\frac{c}{1-c}\right\}$$

giving $0<\delta<1.$ If $1-(1-\delta)\alpha_n,$ then the Mann iteration will be faster than the Picard iteration. Since $0<\delta<1,$ we get that $0<\delta\alpha_n<\alpha_n$ and $1-\alpha_n+\delta\alpha_n<1,$ $\delta\alpha_n<\alpha_n,$ $\delta<1.$ Hence Mann iteration is faster than Picard iteration Recall the following theorem for Mann iteration.

Theorem[2] 5.3: *Let E be a Banach space and K a nonempty closed, convex and bounded subset of $E.$ If $T:K\to K$ is a Lipschitzian strongly pseudocontractive operator such that $F(T)\neq\phi$ then the Mann iteration $\{x_n\}\subset K$ $\alpha_n\in(0,1)$ satisfying*

$$(i) \quad \sum_{n=1}^{\infty} \alpha_n = \infty \qquad (ii) \quad \alpha_n \to 0 \text{ as } n \to \infty$$

converges strongly to the unique fixed point of T.

Here $Q(\alpha_n) = 1 - k^2 \alpha_n$. Putting $\alpha_n = 1$ for Picard iteration we have $Q(1) = 1 - k^2$. Since $\alpha_n < 1$, $1 - k^2 < 1 - k^2 \alpha_n$. Hence Picard iteration is faster than Mann iteration. Let us recall the following theorem for the Ishikawa iteration.

Theorem 5.4: *Let E be an arbitrary Banach space, K a closed, convex subset of E and $T: K \to K$ an operator satisfying conditions $(z_1), (z_2)$ and (z_3). Let $\{x_n\}$ be the Ishikawa iteration where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$.Then $\{x_n\}$ converges strongly to the fixed point of T.*

In this case $Q(\alpha_n) = 1 - (1-\delta)^2 \alpha_n$, for Picard iteration, $\alpha_n = 1$, giving

$$Q(1) = 1 - (1-\delta)^2.$$

Since $\alpha_n < 1$, we have $1 - (1-\delta)^2 < 1 - (1-\delta)^2 \alpha_n$. Hence the Picard iteration is faster than the Ishikawa iteration.

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