ISSN 0974 - 9373

Vol. 21 No. 2 (2017) **Journal of International Academy of Physical Sciences** pp. 97 -106

Almost Hermitian Manifold with Quarter Symmetric Non-Metric Connection

Manisha Kankarej

Zayed University, Dubai, UAE

Email: Manisha.Kankarej@zu.ac.ae

(Received April 24, 2017)

Abstract: In this research, certain results for an almost Hermite manifold equipped with a quarter symmetric non-metric connection has been mentioned. A part from finding the necessary and sufficient condition for a covariant almost analytic vector field with respect to Riemannian connection D to be covariant almost analytic with respect to quarter symmetric non-metric connection ∇, it has also been shown that Nijenhuis tensor with respect to the connection ∇ on an almost Hermite manifold equipped with a quarter symmetric non-metric connection is equal to the Nijenhuis tensor with respect to the Riemannian connection. The geodesics on an almost Hermite manifold equipped with quarter symmetric non-metric connection has also been discussed.

Keywords: Quarter symmetric non-metric connection, almost Hermitian manifold, covariant almost analytic vector field, Nijenhuis tensor, geodesics.

2010 Mathematics Subject Classification: 53B15, 53C55.

1. Introduction

The Riemannian Manifold equipped with a quarter symmetric metric connection has been studied by S. Golab. Yano and Imai, O.C. Andonic, M. C. Chaki and A. Konar studied semi symmetric metric connection. Rastogi continued the systematic study of quarter-symmetric metric connection. Mishra and Pandey studied quarter-symmetric metric connection in Riemannian, Kaehlerian and Sasakian manifolds. S. C. Biswas, U. C. De and J. Sengupta studied quarter-symmetric connection on a SP sasakian manifold. P. N. Pandey and S. K Dubey studied semi symmetric metric connection. S. Bhowmik, A. K. Mondal, U.C. De, A Haseeb, A. Prakash, M. D. Siddiqi studied quarter symmetric non-metric connection.

Definition 1.1. An even dimensional C^∞ differentiable manifold M^n where $n=2m$ is said to be an almost complex manifold (of class C^∞) if there exists a vector valued real linear function F of differentiable class $C^¥$ satisfying

$$F^2(X)+X=0$$

for any vector field $X^{\{F\}}$ is said to give an almost complex structure on M^n. We write $X=FX$.

Definition 1.2. A metric g on an almost complex manifold M^n is said to be a Hermite metric if $g(FX,FY)=g(X,Y)$.

It is always possible to introduce a Hermite metric on an almost complex manifold.

An almost complex manifold M^n endowed with an almost complex structure $\{F\}$ and a Hermite metric g is called an almost Hermite manifold with structure $\{F,g\}$ if

(1.1) $$\bar{X}+X=0,$$

and

(1.2) $$g(FX,FY)=g(X,Y),$$

where F is a tensor of type $(1,1)$, g is a metric tensor and X,Y are arbitrary vector fields.

Definition 1.3. A linear connection D on an n-dimensional C^∞ Riemannian Manifold $\{M^n,g\}$ is said to be a quarter symmetric connection if the torsion tensor S of D satisfies

(1.3) $$S(X,Y)=w(Y)\phi X-w(X)\phi Y,$$

where ϕ is a tensor field of type $(1,1)$ and w is a 1-form associated with vector field ρ

(1.4) $$w(X)=g(X,\rho).$$

If the quarter symmetric connection D satisfies:

(1.5) $\quad (D_X g)(X,Y) = 0$.

then the connection D is said to be quarter symmetric connection, otherwise it is said to be a quarter symmetric non-metric connection.

Definition 1.4. A vector field on a Riemannian manifold $\{M^n, g\}$ is said to be a Killing vector field if:

(1.6) $\quad L_X g = 0$

where L_X denotes the Lie derivative with respect to the vector field X.

Theorem 1.1: *A necessary and sufficient condition that vector field X on a Riemannian Manifold $\{M^n, g\}$ is a Killing vector is that*

(1.7) $\quad g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$

for any vector fields Y and Z. ∇ is the Levi-Civita connection on M^n.

2. Relation between the Riemannian Connection and the Quarter-Symmetric Non-metric Connection

Conversely, we show that a linear connection ∇ in Hermitian manifold is given by

(2.1) $\quad \nabla_X Y = D_X Y - w(X)\phi Y$

determines a quarter-symmetric connection.

Using (2.1) the torsion tensor of the connection ∇ is given by

$$S(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

Let ∇ be the linear connection and D be a Riemannian connection of a Hermite manifold $\{F, g\}$ such that

(2.2) $\quad \nabla_X Y = D_X Y + U(X,Y)$

where U is a tensor of type $(1,1)$. For ∇ be the quarter symmetric connection in M^n, we have

(2.3) $\quad U(X,Y) = \frac{1}{2}\left[S(X,Y) + S'(X,Y) + S'(Y,X)\right]$, where

(2.4) $$g\left(S'(X,Y),Z\right)=g\left(S(X,Y),Y\right).$$

Using (1.3) and (2.3) we obtain

(2.5) $$U(X,Y)=-w(X)\phi Y.$$

Hence a quarter-symmetric connection ∇ in Hermitian manifold is given by

(2.6) $$\nabla_X Y=D_X Y-w(X)\phi Y,$$

(2.7) $$S(X,Y)=w(Y)\phi X-w(X)\phi Y.$$

The above equation shows that the connection ∇ is quarter-symmetric. Also, we have,

(2.8) $$\begin{aligned}(\nabla_X g)(Y,Z) &= Xg(Y,Z)-g(\nabla_X Y,Z)-g(Y,\nabla_X Z)\\ &= g(\nabla_Y X,Z)+g(Y,\nabla_Z X)-g\left(D_X Y+w(X)\phi Y,Z\right)\\ &\quad -g\left(Y,D_X Z+w(X)\phi Z\right)\\ &= -w(Y)g(\phi X,Z)-w(Z)g(Y,\phi X)\\ &= 2w(X)g(\phi Y,Z)\end{aligned}$$

In virue of (2.7) and (2.8) we conclude that ∇ is a quarter-symmetric non-metric connection. This shows that (2.6) is the relation between Riemannian connection and quarter-symmetric non-metric connection in Hermitian manifold.

3. Some theorems on an Almost Hermite Manifold with Connection ∇

Now we define a tensor $B(X,Y)$, such that

(3.1) $$\nabla(X,Y)-D_X Y=B(X,Y)$$

Therefore, equation (2.6) and (3.1) gives

(3.2) $$B(X,Y)=-w(X)\phi Y$$

By virtue of (2.2) and (3.1) we can say that $B(X,Y)=U(X,Y)$.

Let $A(X,Y)$ and $A'(X,Y)$ be the symmetric and skew symmetric parts of the tensor $B(X,Y)$, therefore from equation (1.5) and (2.4), we get

$$B(X,Y)=2A'(X,Y),$$

and $A(X,Y)=0$ if and only if $w(X)\phi Y=-w(Y)\phi X$.

Thus, we have:

Theorem 3.1: *An almost Hermite manifold equipped with a quarter-symmetric non-metric connection ∇ admits the following:*

1 *The torsion tensor S of the connection ∇ is two times of the skew-symmetric part $A'(X,Y)$ of the tensor $B(X,Y)$ i.e. $B(X,Y)=2A'(X,Y)$.*

2 *The symmetric part of the tensor $B(X,Y)$ vanishes if and only if $w(X)\phi Y=-w(Y)\phi X$.*

From equation (2.1)

$$\nabla(X,Y)-D_X Y=-w(X)\phi Y$$

Operating ϕ we get,

(3.3) $$(\nabla_X\phi)(Y)=\nabla_X\phi(Y)-\phi(\nabla_X Y)$$

i.e. $$(\nabla_X\phi)(Y)=(D_X\phi)(Y)-w(Y)\phi X$$

Replacing X and Y by $\bar{X}$ and $\bar{Y}$, we have where $\bar{X}=\phi X$ we have

(3.4) $$(\nabla_{\bar{X}}\phi)(\bar{Y})=(D_{\bar{X}}\phi)(\bar{Y})-w(Y)\phi^2 X$$

Theorem 3.2: *On an almost Hermite manifold, Nijenhuis tensor with respect to the quarter-symmetric non-metric connection is equal to the Nijenhuis tensor with respect to the Riemannian connection.*

The Nijenhuis tensor with respect to quarter-symmetric non-metric connection ∇ is given by

(3.5) $$\tilde{N}(X,Y)=(\nabla_{\bar{X}}\phi)Y-(\nabla_{\bar{Y}}\phi)X-\overline{(\nabla_{\bar{X}}\phi)Y}-\overline{(\nabla_Y\phi)X}$$

Using relation (3.4), we have

(3.6) $$\left(\nabla_{\bar{X}}\phi\right)\left(\bar{Y}\right)=\left(D_{\bar{X}}\phi\right)\left(\bar{Y}\right)-w\left(Y\right)\phi^{2}X$$

Interchanging X and Y in the equation,

(3.7) $$\left(\nabla_{\bar{Y}}\phi\right)\left(\bar{X}\right)=\left(D_{\bar{Y}}\phi\right)\left(\bar{X}\right)-w\left(X\right)\phi^{2}Y$$

Operating ϕ on both sides of equation (3.4) we have

(3.8) $$\overline{\left(\nabla_{X}\phi\right)\left(\bar{Y}\right)}=\overline{\left(D_{X}\phi\right)\left(\bar{Y}\right)}-w\left(X\right)\phi^{2}X$$

Interchanging X and Y in the equation,

(3.9) $$\overline{\left(\nabla_{Y}\phi\right)\left(\bar{X}\right)}=\overline{\left(D_{Y}\phi\right)\left(\bar{X}\right)}-w\left(X\right)\phi^{2}Y$$

By equations (3.6), (3.7), (3.8) and (3.9) we have

(3.10) $$\tilde{N}\left(X,Y\right)=N\left(X,Y\right)$$

Theorem 3.3: *An almost Hermite manifold equipped with a quarter-symmetric non-metric connection* ∇ *admits the following:*

(3.11) $$\begin{cases} a)\quad S'\left(X,Y,\rho\right)=0 \\ b)\quad S'\left(X,Y,Z\right)+S'\left(Y,Z,X\right)+S'\left(Z,X,Y\right)=0 \\ c)\quad S'\left(\bar{X},Y,\bar{Z}\right)+S'\left(Y,\bar{Z},\bar{X}\right)+S'\left(\bar{Z},\bar{X},Y\right)=0 \end{cases}$$

where $S'\left(X,Y,Z\right)$ *is defined as*

(3.12) $$S'\left(X,Y,Z\right)=g\left(S\left(X,Y\right),Z\right)$$

Proof: Using equation (1.3) in the equation (3.12), we get

$$S'\left(X,Y,Z\right)=g\left(S\left(X,Y\right),Z\right),$$

$$S'\left(X,Y,Z\right)=g\left(w\left(Y\right)\phi X-w\left(X\right)\phi Y,Z\right)$$

(3.13) $$S'\left(X,Y,Z\right)=g\left(w\left(Y\right)\phi X,Z\right)-g\left(w\left(X\right)\phi Y,Z\right)$$

Putting $Z=\rho$ in (2.11) and using $w\left(X\right)=g\left(X,\rho\right)$, we have (3.11a),

$$S'\left(X,Y,\rho\right)=g\left(w\left(Y\right)\phi X,\rho\right)-g\left(w\left(X\right)\phi Y,\rho\right)$$

$$S'(X,Y,\rho) = g\left(g(Y,\rho)\phi X,\rho\right) - g\left(g(X,\rho)\phi Y,\rho\right) = 0$$
$$S'(X,Y,\rho) = 0$$

Equation (3.13) can be written as

(3.14) $$S'(Y,Z,X) = g\left(w(Z)\phi Y, X\right) - g\left(w(Y)\phi Z, X\right)$$

and

(3.15) $$S'(Z,X,Y) = g\left(w(X)\phi Z, Y\right) - g\left(w(Z)\phi X, Y\right)$$

Adding (3.13), (3.14) and (3.15), we get (3.11b).

By adding equations (3.13), (3.14) and (3.15) after barring X and Z, we obtain (3.11c).

4. Covariant Almost Analytic Vector Field

In an almost Hermite manifold, if 1-form w satisfies

(4.1) $$w(\nabla_X\phi)(Y) - (\nabla_Y\phi)(X) = (\nabla_{\bar{X}} w)(Y) - (\nabla_Y w)(\bar{X})$$

then the 1-form w is said to be covariant almost analytic vector field.
Subtracting (3.7) from (3.6), we have

(4.2) $$(\nabla_X\phi)(Y) - (\nabla_Y\phi)(X) = (D_X\phi)(Y) - (D_Y\phi)(X) - w(X)\phi^2 Y + w(X)\phi^2 Y$$

From equation (3.4), we may write

(4.3) $$w\left((\nabla_X\phi)(Y) - (\nabla_Y\phi)(X)\right) = w\left((D_X\phi)(Y) - (D_Y\phi)(X)\right) - w\left(w(X)\phi^2 Y + w(X)\phi^2 Y\right)$$

$$\Rightarrow \quad w\left((\nabla_X\phi)(Y) - (\nabla_Y\phi)(X)\right) = w\left((D_X\phi)(Y) - (D_Y\phi)(X)\right) - w\left(w(X)w\left(\phi^2 Y\right) - w(Y)w\left(\phi^2 X\right)\right)$$

Now we define

(4.4) $$(\nabla_X w)Y = g\left(\nabla_Y \rho, Y\right) \text{ and } (D_X w)Y = g\left(D_X \rho, Y\right)$$

Using equations (4.1), (4.3) and (4.4),

$$g(\nabla_X\rho, Y) - g\left(\nabla_Y\rho, \bar{X}\right) = g(D_X\rho, Y) - g\left(D_Y\rho, \bar{X}\right) - \left(w(X)w\left(\phi^2 Y\right) + w(y)w\left(\phi^2 X\right)\right)$$

Then w satisfies

(4.5) $$(\nabla_X w)Y = (D_X w)Y - \left(w(X)w(\phi^2 Y) - w(Y)w(\phi^2 X)\right)$$

Barring X in equation (4.5), we have

(4.6) $$(\nabla_{\bar{X}} w)Y = (D_{\bar{X}} w)Y - \left(w(\bar{X})w(\phi^2 Y) - w(Y)w(\phi^2 \bar{X})\right)$$

Replacing Y by $\bar{Y}$ in equation (4.5), we get

(4.7) $$(\nabla_X w)\bar{Y} = (D_X w)\bar{Y} - \left(w(X)w(\phi^2 \bar{Y}) - w(\bar{Y})w(\phi^2 X)\right)$$

Subtracting (4.7) from (4.6), we obtain

(4.8) $$\begin{aligned}(\nabla_{\bar{X}} w) - (\nabla_X w)\bar{Y} &= (D_{\bar{X}} w)Y - (D_X w)\bar{Y} - \left(w(\bar{X})w(\phi^2 Y) - w(Y)w(\phi^2 \bar{X})\right)\\ &\quad + \left(w(X)w(\phi^2 \bar{Y}) - w(\bar{Y})w(\phi^2 X)\right)\end{aligned}$$

Subtracting (4.8) from (4.3), we have

(4.9) $$\begin{aligned}w\left((\nabla_X \phi)(Y) - (\nabla_Y \phi)(X)\right) - (\nabla_X w)Y + (\nabla_X w)\bar{Y}\\ = w\left((D_X \phi)(Y) - (D_Y \phi)(X)\right) - \left(w(X)w(\phi^2 Y) - w(Y)w(\phi^2 X)\right)\\ - (D_{\bar{X}} w)Y - (D_X w)\bar{Y} + \left(w(\bar{X})w(\phi^2 Y) - w(Y)w(\phi^2 \bar{X})\right)\\ - \left(w(X)w(\phi^2 \bar{Y}) - w(\bar{Y})w(\phi^2 X)\right)\end{aligned}$$

Thus, we may conclude

$$\begin{aligned}&-w(X)w(\phi^2 Y) - w(X)w(\phi^2 \bar{Y}) + w(Y)w(\phi^2 X) - w(Y)w(\phi^2 \bar{X})\\ &+w(\bar{X})w(\phi^2 \bar{Y}) + w(\bar{Y})w(\phi^2 X) = 0\\ &-w(X)\left(w(\phi^2 Y) + w(\phi^2 \bar{Y})\right) + w(Y)\left(w(\phi^2 X) - w(\phi^2 \bar{X})\right)\\ &+w(\bar{X})w(\phi^2 Y) + w(\bar{Y})w(\phi^2 X) = 0\end{aligned}$$

Theorem 4.1: *On an almost Hermite manifold equipped with a quarter-symmetric non-metric connection* ∇, *a covariant almost analytic vector* w *with respect to Riemannian connection* D *is also covariant almost analytic with respect to quarter-symmetric non-metric connection* ∇ *if and only if*

$$\begin{aligned}-w(X)\left(w(\phi^2 Y) + w(\phi^2 \bar{Y})\right) + w(Y)\left(w(\phi^2 X) - w(\phi^2 \bar{X})\right)\\ + w(\bar{X})w(\phi^2 Y) + w(\bar{Y})w(\phi^2 X) = 0\end{aligned}$$

Now, we propose:

Theorem 4.2: *On an almost Hermite manifold equipped with a quarter-symmetric non-metric connection* ∇, *we have*

(4.10) $\tilde{d}w(X,Y) = dw(X,Y)$

where $dw(X,Y) = (D_X w)Y - (D_Y w)X$, $\tilde{d}w(X,Y) = (\nabla_X w)Y - (\nabla_Y w)X$.

Proof: Interchanging X and Y in equation (3.5), we have

(4.11) $(\nabla_Y w)X = (D_Y w)X + aw(\rho)g(X,Y) + bw(X)w(Y)$

Subtracting (4.11) from (4.5), we get (4.10)

In view of (4.10), we conclude:

Corollary 4.1: *In an almost Hermite manifold equipped with a quarter-symmetric non-metric connection* ∇, *if* w *is closed with respect to Riemannian connection* D, *it is also closed with respect to the quarter-symmetric non-metric connection* ∇.

5. Geodesics on an Almost Hermite Manifold

Definition 5.1: Let σ be a curve in M with the tangent field T, vector field Y is σ parallel if and only if $\nabla_T Y = 0$ on σ and the curve is geodesic if and only if $\nabla_T T = 0$ on σ.

Now, we propose:

Theorem 5.1: *In an almost Hermite manifold equipped with quarter-symmetric non-metric connection* ∇, *the Riemannian connection* D *and the quarter-symmetric non-metric connection* ∇ *have same geodesic if and only if* $w(T) = 0$ *or* $\phi T = 0$.

Proof: Putting $X = Y = T$ in equation (2.5), we have

(5.1) $\nabla_T T = D_T T - w(T)\phi T$

From equation (5.1), $D_T T = 0$ implies $\nabla_T T = 0$, if and only if $w(T) = 0$ or $\phi T = 0$.

In view of equation (2.3) and theorem (5.1), we conclude:

Theorem 5.2: *In an almost Hermite manifold equipped with a quarter-symmetric non-metric connection* ∇, *if the Riemannian connection* D *and quarter-symmetric non-metric connection* ∇ *have same geodesics then symmetric part of the tensor* $S(X,Y)$ *vanishes and* $B(X,Y)=A'(X,Y)$.

References

1. S. Golab, On semi-symmetric and quarter-symmetric metric linear connection, *Tensor N. S.*, **29** (1975) 249-254.
2. K.Yano and T. Imai, On semi symmetric metric F-connection, *Tensor N. S.*, **29** (1975) 134-138.
3. O. C. Andonie, Sur une connexion semi-symmetrique lasse invariant to tenseur de Bochner, *Ann. Fac.Sc.Kuishaa Zaire*, **1** (1976) 247-253.
4. S. C. Rastogi, On quarter-symmetric metric connection, *C. R. Acad Sci, Bulgar*, **31** (1978) 811-814.
5. R. S. Mishra and S. N. Pandey, On quarter-symmetric metric F-connection, *Tensor N. S.*, **34(1**) (1980) 1-7.
6. M. C. Chaki and A. Konar, On a type of semi-symmetric connection on a Riemannian manifold, *J. Pure Math*, **1** (1981) 77-80.
7. R. S. Mishra, *Structures on differentiable manifold and their applications*, Chandrama Prakashan 50-A Balrampur House, Allahabad, India, 1984.
8. U. C. De, On a type of semi-symmetric metric connection on a Riemannian manifold, *An. Stiift. Univ. "Al. I. Cuzza" Lasi Sect.I a Math*, **38** (1991) 105-108.
9. U. C. De and S. C. Biswas, On a type of semi-symmetric connection on a Riemannian manifold, *Publ. Del'Institue Mathematique*, **61** (1997) 90-96.
10. S. C. Biswas and U. C De, Quarter-symmetric metric connection in an SP-Sasakian manifold, *Commun. Fac. Sci. Univ. Al*, **46** (1997) 49-56.
11. U. C. De and J. Sengupta, Quarter-symmetric metric connection on a Sasakian manifold, *Commun. Fac. Sci. Univ. Ank. Al*, **49** (2000) 7-13.
12. P. N. Pandey and S. K. Dubey, Almost Grayan manifold admitting semi-symmetric metric connection, *Tensor N. S.*, **65** (2004) 144-152.
13. P. N. Pandey and B. B. Chaturvedi, Almost Hermitian Manifold with Semi-Symmetric Recurrent Connection, *Journal of International Academy of Physical Sciences*, **10** (2006) 69-74.
14. S. Bhowmik, Some properties of a quarter symmetric non-metric connection on a Kahler Manifold, *Bulletin of Kerala Mathematics Association*, **6(1)** (2010) 99-109.
15. A. K. Mondal and U.C. De, Quarter-symmetric non-metric connection on P-sasakian manifolds, *ISRN geometry*, (2012) 1-14.
16. A. Haseeb, A. Prakash, M. D. Siddiqi, On a quarter-symmetric non-metric connection in an epsilon-Lorentzian para-sasakian manifold, *Acta Mathematics Universitatis Comenianae*, **86(1)** (2017) 143-152.