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Doubly Twisted Product Contact CR-Submanifolds in Quasi-Sasakian Manifolds

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Abstract: In current paper, we have shown that a doubly twisted product Contact CR-submanifolds ${}_fN_T \times {}_bN_\perp$ of quasi-Sasakian manifold such that the structure vector field ξ is tangential to $N_\perp$, does not exist. We have also shown that a doubly twisted product Contact CR-submanifolds ${}_fN_T \times {}_bN_\perp$ of Sasakian or cosympletic manifold such that the structure vector field ξ is tangential to $N_\perp$, does not exist.

Keywords: Quasi-Sasakian manifold, contact CR-submanifold, doubly warped and doubly twisted products.

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1. Introduction

There is substantial contribution towards twisted and warped product structures in geometry as well as in Physics. These structures in Riemannian geometry have been an area of interest for various authors[1,2]. Twisted product metric tensors as generalization of warped product metric tensors have enriched the study of submanifold theory. In 1967, D. E. Blair[3] introduced the notion of quasi-Sasakian manifold to unify Sasakian and cosympletic manifolds and in 1977, S. Kanemaki[4] studied quasi-Sasakian manifolds.

K. A. Khan and V. A. Khan[5] showed the non-existence of warped product contact CR-submanifold of a trans-Sasakian manifold while S. Uddin[7] showed the non-existence doubly of warped and doubly twisted product submanifolds of a nearly Kaehler manifold. In this paper we showed that the non-existence of doubly twisted product contact CR-submanifold of a quasi-Sasakian manifold.

2. Preliminaries

Let $\bar{M}$ be a $(2n+1)-$dimensional almost contact metric manifold with almost contact metric structure (ϕ,ξ,η,g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a $1-$form and g is a compatible Riemannian metric on $\bar{M}$ such that

(2.1) $$\phi^2 = -I + \eta\otimes\xi, \quad \eta(\xi)=1, \ \phi\xi = 0,$$

(2.2) $$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

(2.3) $$g(\phi X,Y) = g(X,\phi Y), \quad g(X,\xi)=\eta(X),$$

where $X,Y \in T\bar{M}$.

An almost contact metric manifold is said to be contact manifold if

(2.4) $$d\eta(X,Y) = \varphi(X,Y) = g(X,\phi Y),$$

$\varphi(X,Y)$ is being called fundamental $2-$form of $\bar{M}$.

If the characteristic vector field ξ is a killing vector, then the contact manifold is called a K-contact manifold. A contact metric manifold is K-contact, if and only if, $\bar{\nabla}_X\xi = -\phi X$, for any X on $\bar{M}$. On the other hand, a normal contact metric manifold is known as Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

(2.5) $$(\bar{\nabla}_X\phi)Y = g(X,Y)\xi - \eta(Y)X,$$

for any vector field X,Y. An almost contact metric structure is called quasi-Sasakian if it is normal and its fundamental form is closed, that is, for every $X,Y \in T\bar{M}$.

(2.6) $$[\phi,\phi](X,Y) + d\eta(X,Y)\xi = 0,$$

(2.7) $$d\varphi = 0, \ \varphi(X,Y) = g(X,\phi Y).$$

This was first introduced by Blair [3]. There are many types of quasi-Sasakian structure ranging from the cosymplectic case, $d\eta = 0,$ (rank $\eta = 0,$), to the Sasakian case, $\eta \wedge (d\eta)^p \neq 0$ (rank $\eta = 2n+1, \varphi = d\eta$). The 1-form $\eta,$ has rank $r' = 2p$ if $(d\eta)^p \neq 0$ and $\eta \wedge (d\eta)^p \neq 0$ and has rank $r = 2n+1,$ if $(d\eta)^p = 0$ and $\eta \wedge (d\eta)^p = 0$. We also say that r' is the rank of the quasi-Sasakian structure. Blair[3] also proved that there are no quasi-Sasakian manifold of even rank. In order to study the properties of quasi-Sasakian manifolds Blair[3] proved some theorems regarding Kaehlerian manifolds and the existence of quasi-Sasakian manifolds. The fundamental vector field ξ of a quasi-Sasakian structure is a killing vector field that is, $\text{Ł}_{\xi} g = 0.$ For quasi-Sasakian manifolds Blair[3] proved that

(2.8) $$\bar{\nabla}_X \xi = -\frac{1}{2}\phi X,$$

for any vector field X.

Let M be a submanifold immersed in $\bar{M}$ with induced metric g and $\bar{\nabla}$ and ∇ the Levi-Civita connections on $\bar{M}$ and M respectively. Then the Gauss and Weingarten formulae are given by

(2.9) $$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

and

(2.10) $$\bar{\nabla}_X V = -A_V X + \nabla_X^{\perp} V$$

for all $X, Y \in TM$ and $V \in T^{\perp}M$, where $\nabla^{\perp}$ is the connection on $T^{\perp}M$, h and A_V are second fundamental form and Weingarten map associated with V as

(2.11) $$g(A_V X, Y) = g(h(X,Y), V).$$

For any $X \in TM$, we write

(2.12) $$\phi X = PX + FX,$$

where PX and FX denote the tangential component and the normal component of ϕX.

Similarly, for $V \in TM^{\perp}$, we have

(2.13) $$\phi V = tV + wV,$$

where tV is the tangential component and wV is the normal component of ϕV.

Definition 2.1: Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds of dimension n_1 and n_2, respectively and $\pi : N_1 \times N_2 \to N_1$ and $\pi : N_1 \times N_2 \to N_2$ be the canonical projections. Let $b : N_1 \times N_2 \to (0, \infty)$ and $f : N_1 \times N_2 \to (0, \infty)$ be two smooth functions. The doubly twisted product of (N_1, g_1) and (N_2, g_2) with twisting functions b and f is defined to be the product manifold ${}_f N_1 \times_b N_2$, with the metric tensor $g = f^2 \pi^* g_1 \oplus b^2 \sigma^* g_2$. In particular if $f = 1$, then $N_1 \times_b N_2$ is called the twisted product of (N_1, g_1) and (N_2, g_2) with twisting function b. Moreover, if b only depends on the points of N_1, then $N_1 \times_b N_2$ is called the warped product of (N_1, g_1) and (N_2, g_2) with warping function b. As a generalization of the warped product of two Riemannian manifolds, ${}_f N_1 \times_b N_2$ is called the doubly warped product of Riemannian manifolds (N_1, g_1) and (N_2, g_2) with warping functions b and f if b and f only depend on the points of N_1 and N_2 respectively[1].

Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds with Levi-Civita connections ∇^1 and ∇^2 respectively and let ∇ denote the Levi-Civita connection of the doubly twisted product ${}_f N_1 \times_b N_2$ of (N_1, g_1) and (N_2, g_2) with twisting functions b and f. Then we have the following Proposition[1].

Proposition 2.1: *If $X, Y \in TN_1$ and $V \in TN_2$, then*

(2.14) $$\nabla_X Y = \nabla^{\perp}_X Y + X \log(f) Y + Y \log(f) X - g(X, Y) \nabla \log(f),$$

(2.15) $$\nabla_X V = V \log(f) X + X \log(b) V.$$

Definition 2.2: A submanifold M tangent to ξ, is called contact CR-submanifold if there exists on M a differential distribution $D : x \to D_x \subset T_x M$ such that (i) D_x is invariant under ϕ i.e $\phi D_x \subset D_x$ for each $x \in M$, (ii) the orthogonal complementary distribution $D^{\perp} : x \to D^{\perp} \subset T_x M$ of the distribution

D on M is totally real, i.e $\phi D^{\perp} \subset T_x^{\perp} M$ (iii) $TM = D \oplus D^{\perp} \oplus \{\xi\}$, where $T_x M$, $T_x^{\perp} M$ are the tangent space and normal space of M at x respectively and $\oplus$ denotes the orthogonal direct sum.

If N_T and $N_{\perp}$ are invariant and anti-invariant submanifolds of quasi-Sasakian manifold $\bar{M}$ respectively, then their doubly twisted product is ${}_f N_T \times {}_b N_{\perp}$.

3. Doubly Twisted Product Contact CR-Submanifolds of a Quasi-Sasakian Manifold

In this section, we assume that $\bar{M}$ is a quasi-Sasakian manifold and ${}_f N_T \times {}_b N_{\perp}$ be a doubly twisted product contact CR-submanifold of $\bar{M}$.

Theorem 3.1: *A doubly twisted product contact CR-submanifolds ${}_f N_T \times {}_b N_{\perp}$ of $\bar{M}$ such that the structure vector field ξ is tangential to $N_{\perp}$, does not exist.*

Proof*:* Let $M = {}_f N_T \times {}_b N_{\perp}$ is a doubly twisted product contact CR-submanifold of a quasi-Sasakian manifold $\bar{M}$. Then by using equations (2.8), (2.9) and (2.15) of Proposition 2.3, we have

$$\bar{\nabla}_X \xi = \nabla_X \xi + h(X, \xi),$$

$$-\frac{1}{2}\phi X = X \log(b)\xi + \xi \log(f) X + h(X, \xi), \tag{3.2}$$

for any $X \in TN_T$. Multiplying both sides of equation (3.2) by ξ, we get $X \log(b) = 0$, i.e b constant on N_T. In the same way, using equations (2.8), (2.9) and (2.14) of Proposition 2.3, we have

$$\bar{\nabla}_U \xi = \nabla_U \xi + h(U, \xi),$$

$$-\frac{1}{2}\phi U = \nabla_U^{\perp} \xi + U \log(b)\xi + \xi \log(b) U - \eta(X) \nabla \log(b) + h(U, \xi),$$

for any $U \in TN_{\perp}$, where $\nabla^{\perp}$ denote the Levi-Civita connection on $N_{\perp}$. Multiplying both sides of equation (3.1) by ξ, we get

$$-\frac{1}{2}g(\phi U,\xi) = U\log(b) + \eta(U)\xi\log(b) - \eta(U)\xi\log(b),$$
$$0 = U\log(b),$$

which implies that b is constant on $N_\perp$.

Corollary 3.1: *A doubly twisted product contact CR-submanifolds* ${}_fN_T \times {}_bN_\perp$ *of Sasakian manifold such that the structure vector field* ξ *is tangential to* $N_\perp$, *does not exist.*

Corollary 3.2: *A doubly twisted product contact CR-submanifolds* ${}_fN_T \times {}_bN_\perp$ *of cosympletic manifold such that the structure vector field* ξ *is tangential to* $N_\perp$, *does not exist.*

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