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# On Directional Recurrence in Cartan Sense

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**Abstract:** Shivalika Saxena[7] introduced the concept of directional recurrence of Berwald curvature tensor, which is a generalization of recurrence Berwald curvature tensor and obtained several results for such a space. In the present paper, the concept of directional recurrence of Cartan curvature tensor is being introduced and certain properties of such space are obtained.

**Keywords:** Finsler space, directional recurrence, Cartan curvature tensor.

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## 1. Introduction

In 1949, H. S. Ruse[1] introduced a 3-dimensional Riemannian space whose curvature tensor is recurrent in every direction. Such Riemannian space was named as Riemannian space with recurrent curvature. The concept was extended to an n-dimensional Riemannian space by A. G. Walker[2]which was further extended to non-Riemannian spaces and to Finsler spaces by several authors including A. Moor[3]. Significant contribution were made by several Indian Finsler geometers including R. S. Mishra and H. D. Pande[4], R. N. Shen[5], U. P. Singh, B. N. Prasad, R. B. Mishra, P. N. Pandey[6], H. S. Shukla and T. N. Pandey in this field. Shivalika Saxena[7] introduced the concept of directional recurrence of Berwald curvature tensor, which is a generalization of recurrence Berwald curvature tensor and obtained several results for such a space.

In the present paper, the concept of directional recurrence of Cartan curvature tensor is being introduced and certain properties of such space are obtained.

## 2. Preliminaries

Suppose $F_n$ be an $n-$dimensional Finsler space having $F$ as a metric function satisfying the requisite conditions[8], $g_{ij}$ is corresponding metric tensor, $G^i_{jk}$ Berwald connection coefficients and $\Gamma^{*i}_{jk}$ Cartan connection coefficients.

Cartan covariant derivative of an arbitrary tensor field $T^i_j$ is given by

(2.1) $$T^i_{j|k} = \partial_k T^i_j - (\dot{\partial}_r T^i_j) G^r_k + T^r_j \Gamma^{*i}_{rk} - T^i_r \Gamma^{*r}_{jk}.$$

The commutation formulae between Cartan covariant differentiation with respect to $x^k$ and partial differentiation with respect to $y^j$ are given by

(2.2) $$\dot{\partial}_j (X^i_{|k}) - (\dot{\partial}_j X^i)_{|k} = X^r (\dot{\partial}_j \Gamma^{*i}_{rk}) - (\dot{\partial}_r X^i)(\dot{\partial}_j \Gamma^{*r}_{sk}) y^s.$$

Berwald defined the curvature tensor $H^i_{jkh}$ whichsatisfies the following conditions

(2.3) (a) $\dot{\partial}_j H^i_{kh} = H^i_{jkh}$ (b) $\dot{x}^j H^i_{jkh} = H^i_{kh}$

(c) $\dot{x}^k H^i_{kh} = H^j_h$ (d) $H^r_{jr} = H_j$

(e) $H^r_r = \dot{x}^j H_j = (n-1)H$ (f) $\dot{x}^k H^i_k = 0.$

Cartan defined the curvature tensor $K^i_{jkh}$ whichsatisfies the following conditions

(2.4) (a) $\dot{x}^j K^i_{jkh} = H^i_{kh}$ (b) $K^r_{jkr} = K_{jk}$,

where $K_{jk}$ be Cartan Ricci tensor.

The projective deviation tensor $W^i_h$ of a Finsler space is defined as

(2.5) $$W^i_h = H^i_h - H\delta^i_h - \frac{y^i}{n+1}(\dot{\partial}_r H^r_h - \dot{\partial}_h H).$$

**Definition 2.1:**A vector field $v^i(x^j)$ in a Finsler space $F_n$ is called contra, concurrent, special concircular and torse forming vector fields respectively, ifit satisfies the following conditions

(2.6) (a) $v^i_{|k} = 0$ (b) $v^i_{|k} = c\delta^i_k$

(c) $v^i_{|k} = \rho\delta^i_k$ (d) $v^i_{|k} = \rho\delta^i_k + \mu_k v^i$,

where $c$ is a nonzero constant, $\rho$ is a scalar field and $\mu_k$ is a nonzero covariant vector field.

## 3. Directional Recurrence of Geometric Objects in Berwald Sense and Cartan Sense

Let $\Omega$ be any geometric object and $v^i$ be the components of a contravariant vector field in a Finsler space. Shivalika Saxena[7] calls the geometric object $\Omega$ recurrent in the direction $v^i$ if there exists a non-null scalar field $\phi$ such that

(3.1) $\Omega_{(k)} v^k = \phi\Omega.$.

We shall call it directional recurrence in the sense of Berwald.

We define a directional recurrence of the geometric object $\Omega$ in the direction by $v^i$ by

(3.2) $\Omega_{|k} v^k = \phi\Omega,$

and call it the directional derivative of the geometric object $\Omega$ in the direction $v^i$, in the sense of Eli Cartan.

Directional derivatives in the sense of Berwald and Eli Cartan are in general, different. However, if the geometrical object $\Omega$ is a scalar field, both directional derivatives coincide. This can be seen with the help of definitions of Berwald covariant derivative and Cartan covariant derivative of a scalar field.

If $T^i_j$ are components of a tensor field, then its directional derivative in the direction $v^i$ in Berwald sense and Cartan sense are given by

$$T^i_{j(k)} v^k = v^k [\partial_k T^i_j - (\dot{\partial}_r T^i_j) G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}]$$

and

$$T^i_{j|k} v^k = v^k [\partial_k T^i_j - (\dot{\partial}_r T^i_j) G^r_k + T^r_j \Gamma^{*i}_{rk} - T^i_r \Gamma^{*r}_{jk}],$$

respectively for $G^i_{jk} \neq \Gamma^{*i}_{jk}$ in general. However, $G^i_{jk} = \Gamma^{*i}_{jk}$ if the space is a Landsberg space. Therefore, in a Landsberge space, both directional derivatives coincide.

A geometric object $\Omega$ is called recurrent in the sense of Eli Cartan if there exists a non-zero vector field $\lambda_m$ such that

(3.3) $\Omega_{|m} = \lambda_m \Omega.$

Transvecting (3.3) with $v^m$ and putting $\lambda_m v^m = \phi,$ we get (3.2). This shows that a geometrical object which is recurrent, is recurrent in the direction $v^i$. Its converse is not necessarily true.

A contra vector field $v^i$ characterized by

(3.4) $v^i_{|k} = 0,$

is not recurrent in any direction.

A concurrent vector field is characterized by

(3.5) $v^i_{|k} = c\delta^i_k.$

Tansvecting equation (3.5) with $v^k$, we get

(3.6) $v^i_{|k} v^k = cv^i$ .

Transvecting equation (3.5) with $w^k$ , we get

(3.7) $v^i_{|k} w^k = cw^i$ .

From equations (3.6) and (3.7), we conclude that

**Theorem3.1:***A concurrentvector fieldin a Finsler space is recurrent in its own direction but not recurrent in the direction of any other vector.*

Transvecting equation (2.2 c) with $v^k$ , we get

(3.8) $v^i_{|k} v^k = \rho v^i$ .

Again transvecting (2.2 c) with $w^k$ , we obtain

(3.9) $v^i_{|k} w^k = \rho w^i$ .

This leads to

**Theorem 3.2:***A special concircular vector field in a Finsler space is recurrent in its own direction but not recurrent in the direction of any other vector.*

Transvecting equation (2.2 d) with $v^k$ , we get

$v^i_{|k} v^k = \rho v^i + \mu_k v^k v^i$ ,

which implies

(3.10) $v^i_{|k} v^k = (\rho + \mu_k v^k) v^i$ .

Again transvecting equation (2.2 d) with $w^k$ , we get

(3.11) $v^i_{|k} w^k = \rho w^i + (\mu_k w^k) v^i$ .

From equations (3.10) and (3.11), we conclude that

**Theorem 3.3:** *A torse forming vector field in a Finsler space is recurrent in its own direction but not recurrent in the direction of any other vector.*

Cartan covariant derivative of the metric tensor $g_{ij}$ is given by

(3.12) $g_{ij|k} = 0$ .

This shows that the metric tensor $g_{ij}$ is not recurrent in any direction.

## 4. Directional Recurrence of Cartan Curvature Tensor

Let us consider a Finsler space whose Cartan curvature tensor $K^i_{jkh}$ is recurrent in the direction $v^i$. Such space is characterized by

(4.1) $$K^i_{jkh|m} v^m = \phi K^i_{jkh} ,$$

where the scalar field $\phi \neq 0$.

Transvecting equation (4.1) with $y^j$ and using equation (2.4 b), we get

(4.2) $$H^i_{kh|m} v^m = \phi H^i_{kh} .$$

Transvecting equation (4.2) with $y^k$ and using equation (2.4 c), we have

(4.3) $$H^i_{h|m} v^m = \phi H^i_h .$$

Contracting the indices $i$ and $h$ in equation (4.1), (4.2) and (4.3) and using equations (2.4 b), (2.3 d) and (2.3 e) respectively, we obtain

(4.4) $$K_{jk|m} v^m = \phi K_{jk} .$$

(4.5) $$H_{k|m} v^m = \phi H_k .$$

(4.6) $$H_{|m} v^m = \phi H .$$

From equation (4.2), (4.3), (4.4), (4.5) and (4.6), we conclude that:

**Theorem 4.1:** *Cartan Ricci tensor $K_{jk}$, Berwald torsion tensor $H^i_{kh}$, Berwald deviation tensor $H^i_k$, the vector $H_k$ and the scalar curvature $H$ of a Finsler space which is recurrent in the direction $v^i$, in Cartan sense, are necessarily recurrent in the direction $v^i$.*

Differentiating (4.5) partially with respect to $y^j$, we get

(4.7) $$v^m \dot{\partial}_j H_{k|m} = (\dot{\partial}_j \phi) H_k + \phi (\dot{\partial}_j H_k) .$$

Using the commutative formula exhibited by (2.2), we get

(4.8) $$v^m [(\dot{\partial}_j H_k)_{|m} - H_r \dot{\partial}_j \Gamma^{*r}_{km} - (\dot{\partial}_r H_k)(\dot{\partial}_j \Gamma^{*r}_{sm}) y^s] = (\dot{\partial}_j \phi) H_k + \phi (\dot{\partial}_j H_k),$$

which implies

(4.9) $$H_{jk|m} v^m - v^m H_r \dot{\partial}_j \Gamma^{*r}_{km} - v^m H_{rk} (\dot{\partial}_j \Gamma^{*r}_{sm}) y^s = (\dot{\partial}_j \phi) H_k + \phi H_{jk} .$$

We may write (4.9) as

(4.10) $$H_{jk|m} v^m - \phi H_{jk} = v^m H_r \dot{\partial}_j \Gamma^{*r}_{km} - v^m H_{rk} (\dot{\partial}_j \Gamma^{*r}_{sm}) y^s + (\dot{\partial}_j \phi) H_k .$$

This leads to

**Theorem 4.2:** *Let a Finsler space be recurrent in the direction of the vector field* $v^i$ *, in Cartan sense. Then the Berwald Ricci tensor* $H_{jk}$ *is recurrent in the direction* $v^i$ *if and only if there holds the condition*

$$v^m H_r \dot{\partial}_j \Gamma^{*r}_{km} - v^m H_{rk} (\dot{\partial}_j \Gamma^{*r}_{sm}) y^s + (\dot{\partial}_j \phi) H_k = 0. \tag{4.11}$$

Differentiating (4.2) partially with respect to $y^j$ , we get

$$v^m \dot{\partial}_j H^i_{kh|m} = (\dot{\partial}_j \phi) H^i_{kh} + \phi (\dot{\partial}_j H^i_{kh}). \tag{4.12}$$

Using the commutation formula (2.2) in equation (4.12), we get

$$(4.13) \qquad v^m [(\dot{\partial}_j H^i_{kh})_{|m} + H^r_{kh} \dot{\partial}_j \Gamma^{*i}_{rm} - H^i_{rh} \dot{\partial}_j \Gamma^{*r}_{km} - H^i_{kr} \dot{\partial}_j \Gamma^{*r}_{hm}$$
$$- (\dot{\partial}_r H^i_{kh})(\dot{\partial}_j \Gamma^{*r}_{sm}) y^s] = (\dot{\partial}_j \phi) H^i_{kh} + \phi \dot{\partial}_j H^i_{kh}).$$

In view of (2.3 a), (4.13) may be written as

$$(4.14) \qquad H^i_{jkh|m} v^m - \phi H^i_{jkh} = (\dot{\partial}_j \phi) H^i_{kh} + v^m [H^i_{rh} \dot{\partial}_j \Gamma^{*r}_{km} + H^i_{kr} \dot{\partial}_j \Gamma^{*r}_{hm}$$
$$+ H^i_{rkh} \dot{\partial}_j \Gamma^{*r}_{sm} y^s - H^r_{kh} \dot{\partial}_j \Gamma^{*i}_{rm}].$$

From equation (4.14), we conclude that

**Theorem 4.3:** *Let a Finsler space be recurrent in the direction of the vector field* $v^i$ *, in Cartan sense. Then its Berwald curvature tensor is recurrent in the direction* $v^i$ *, in Cartan sense, if and only if the following condition holds*

$$(\dot{\partial}_j \phi) H^i_{kh} + v^m [H^i_{rh} \dot{\partial}_j \Gamma^{*r}_{km} + H^i_{kr} \dot{\partial}_j \Gamma^{*r}_{hm} + H^i_{rkh} \dot{\partial}_j \Gamma^{*r}_{sm} y^s - H^r_{kh} \dot{\partial}_j \Gamma^{*i}_{rm}] = 0. \tag{4.15}$$

Covariant differentiation of (2.5) with respect to $x^m$ in the sense of Cartan, we have

$$W^i_{h|m} = H^i_{h|m} - H_{|m} \delta^i_h - \frac{y^i}{n+1} (\dot{\partial}_r H^r_h - \dot{\partial}_h H)_{|m}. \tag{4.17}$$

Transvecting equation (4.17) with $v^m$ , we get

$$v^m W^i_{h|m} = v^m H^i_{h|m} - v^m H_{|m} \delta^i_h - \frac{y^i}{(n+1)} [(\dot{\partial}_r H^r_h)_{|m} - (\dot{\partial}_h H)_{|m}] v^m. \tag{4.18}$$

Using the commutation formula for Cartan covariant differentiation and directional differentiation, we have

$$\dot{\partial}_k (H^i_{h|m}) - (\dot{\partial}_k H^i_h)_{|m} = H^r_h \dot{\partial}_k \Gamma^{*i}_{hm} - H^i_r \dot{\partial}_k \Gamma^{*r}_{hm} - (\dot{\partial}_r H^i_h) \dot{\partial}_k \Gamma^{*r}_{sm} y^s$$

and $\qquad \dot{\partial}_k(H_{|m}) - (\dot{\partial}_k H)_{|m} = -(\dot{\partial}_r H)\dot{\partial}_k \Gamma^{*r}_{sm} y^s$ .

Transvecting these equation with $v^m$ and using (4.3) and (4.6), we get

(4.19) $(\dot{\partial}_k H^i_h)_{|m} v^m = \phi \dot{\partial}_k H^i_h + H^i_h \dot{\partial}_k \phi - H^r_h \dot{\partial}_k \Gamma^{*i}_{rm} v^m$
$+ H^i_r \dot{\partial}_k \Gamma^{*r}_{hm} w^m + (\dot{\partial}_r H^i_h)\dot{\partial}_k \Gamma^{*r}_{sm} y^s v^m$

and

(4.20) $\qquad \dot{\partial}_k(H)_{|m} v^m = \phi \dot{\partial}_k H + H \dot{\partial}_k \phi + (\dot{\partial}_r H)\dot{\partial}_k \Gamma^{*r}_{sm} y^s v^m$.

Contracting the indices $k$ and $i$ in (4.19), we have

(4.21) $\qquad (\dot{\partial}_r H^r_h)_{|m} v^m = \phi \dot{\partial}_r H^r_h + H^r_h \dot{\partial}_r \phi - H^t_h \dot{\partial}_r \Gamma^{*i}_{tm} v^m$
$+ H^i_t \dot{\partial}_k \Gamma^{*t}_{hm} w^m + (\dot{\partial}_t H^r_h)\dot{\partial}_r \Gamma^{*t}_{sm} y^s v^m$.

From (4.20) and (4.21), we get

(4.22) $\qquad [(\dot{\partial}_r H^r_h)_{|m} - (\dot{\partial}_h H)_{|m}]v^m = \phi(\dot{\partial}_r H^r_h - \dot{\partial}_h H) + H^r_h(\dot{\partial}_r \phi) - H(\dot{\partial}_h \phi)$
$+ v^m H^r_s \dot{\partial}_r \Gamma^{*s}_{hm} - v^m H^s_h \dot{\partial}_r \Gamma^{*r}_{sm} + v^m(\dot{\partial}_s H^r_h)$
$(\dot{\partial}_r \Gamma^{*s}_{lm}) y^l - v^m(\dot{\partial}_r H)(\dot{\partial}_h \Gamma^{*r}_{sm}) y^s$.

Using (4.22) in equation (4.18), we find

(4.23) $\qquad v^m W^i_{h|m} = \phi W^i_h - \dfrac{y^i}{(n+1)}[H^r_h \dot{\partial}_r \phi - H \dot{\partial}_h \phi + v^m H^r_s \dot{\partial}_r \Gamma^{*s}_{hm}$

$- v^m H^s_h \dot{\partial}_r \Gamma^{*r}_{sm} + v^m(\dot{\partial}_s H^r_h)(\dot{\partial}_r \Gamma^{*s}_{lm}) y^l - v^m(\dot{\partial}_r H)(\dot{\partial}_h \Gamma^{*r}_{sm}) y^s]$.

This leads to

**Theorem 4.4**:*If a Finsler space is recurrent in the direction $v^i$ in Cartan sense, then the projective deviation tensor of the Finsler space is recurrent in that direction if the following condition is satisfied*

(4.24) $\qquad [H^r_h \dot{\partial}_r \phi - H \dot{\partial}_h \phi + v^m H^r_s \dot{\partial}_r \Gamma^{*s}_{hm} - v^m H^s_h \dot{\partial}_r \Gamma^{*r}_{sm}$
$+ v^m(\dot{\partial}_s H^r_h)(\dot{\partial}_r \Gamma^{*s}_{lm}) y^l - v^m(\dot{\partial}_r H)(\dot{\partial}_h \Gamma^{*r}_{sm}) y^s] = 0.$

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