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# Continuous Fractional Wavelet Transform

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**Abstract:** In this paper the continuous fractional Wavelet transform is introduced and exploiting the theory of fractional Fourier transform, Parseval's formula and inversion formula for the fractional Wavelet transform are developed.

**Keywords:** Fourier transform, Fractional Fourier transform, Wavelet transform, Fractional Wavelet transform.

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## 1. Introduction

The continuous Wavelet transform played an important role in many problems of Mathematics, Physics and engineering sciences. Many researchers in different disciplines are working in this area.

The characterization of continuous Wavelet transform were extensively studied[2-3] in 1992 and 1995. Motivated from the work[1], our main objective in this paper is to develop the theory of fractional Wavelet transform, which will be helpful in the study of quantum mechanics, signal processing and other areas of engineering sciences.

The following definitions, formulae and properties of Wavelet transform, which are useful in our present paper.

**Definition 1.1:** *A function* $\psi$ *is called wavelet if it satisfies the following conditions;*

*(i)* $\psi \in L^2(\mathbb{R})$,

*(ii) It satisfies the admissibility condition*

$$(1.1)\qquad C_\psi \equiv \int_{-\infty}^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty .$$

**Definition 1.2:** *If* $\psi \in L^2(\mathbb{R})$, *then the continuous transformation* $W_\psi$ *of function* $f \in L^2(\mathbb{R})$ *is defined by*

$$(1.2)\qquad W_\psi[f](a,b) = \langle f, \psi_{a,b} \rangle = \int_{-\infty}^{+\infty} f(t)\overline{\psi_{a,b}(t)}dt .$$

The relation between Fourier transform and wavelet transform is given by

$$(1.3)\qquad F\{W_\psi[f](a,b)\} = \sqrt{|a|}\,\hat{f}(\omega)\overline{\hat{\psi}}(a\omega) .$$

**Definition 1.3:** *The fractional Fourier transform of order* $\alpha (0 < \alpha \le 1)$ *for a function* $\phi \in L^2(\mathbb{R})$ *is defined as*

$$(1.4)\qquad \hat{\phi}_\alpha(\omega) = \int_{-\infty}^{+\infty} e^{-i(sign\,\omega)|\omega|^{\frac{1}{\alpha}}t}\phi(t)dt, \omega \in \mathbb{R}, t \in \mathbb{R}$$

*and corresponding inverse fractional Fourier transform is*

$$(1.5)\qquad \phi(t) = \int_{-\infty}^{+\infty} e^{i(sign\,\omega)|\omega|^{\frac{1}{\alpha}}t}\hat{\phi}_\alpha(\omega)|\omega|^{\frac{1}{\alpha}-1} d\omega .$$

The whole paper is organized as follows;

Section 1 is introductory, various definitions, properties and formulae are given. In section 2 the definitions of fractional Wavelet, continuous fractional Wavelet transform, Parseval’s formula and inversion formula for the fractional Wavelet transform are introduced and relation between fractional Wavelet transform and fractional Fourier transform obtained.

## 2. Continuous Fractional Wavelet Transform

In this section, the continuous fractional Wavelet transform is defined and Parseval's formula together with inversion formula of fractional Wavelet transform obtained.

**Theorem 2.1:** *If* $f,\ g \in L^2(\mathbb{R})$, *then Parseval's formula for the fractional Fourier transform isgiven by*

*(2.1)* $$< f, g >= \frac{1}{2\pi\alpha} <|\omega|^{\frac{1}{\alpha}-1}\ \hat{f}_\alpha(\omega), \hat{g}_\alpha(\omega)>.$$

***Proof:*** Let $f, g \in L^2(\mathbb{R})$, we take

$$< f, g >= \int_{-\infty}^{+\infty} f(t)\overline{g(t)}dt$$

$$= \frac{1}{2\pi\alpha}\int_{-\infty}^{+\infty} f(t)\left(\overline{\int_{-\infty}^{+\infty} e^{i(sign\omega)|\omega|^{\frac{1}{\alpha}}t}\,|\omega|^{\frac{1}{\alpha}-1}\ \hat{g}_\alpha(\omega)d\omega}\right)dt$$

$$= \frac{1}{2\pi\alpha}\int_{-\infty}^{+\infty} f(t)\left(\int_{-\infty}^{+\infty} e^{-i(sign\omega)|\omega|^{\frac{1}{\alpha}}t}\,|\omega|^{\frac{1}{\alpha}-1}\ \overline{\hat{g}_\alpha(\omega)}d\omega\right)dt.$$

Applying Fubini's theorem, we have

$$< f, g >= \frac{1}{2\pi\alpha}\int_{-\infty}^{+\infty} e^{-i(sign\omega)|\omega|^{\frac{1}{\alpha}}t} f(t)\left(\int_{-\infty}^{+\infty} |\omega|^{\frac{1}{\alpha}-1}\ \overline{\hat{g}_\alpha(\omega)}d\omega\right)dt$$

$$= \frac{1}{2\pi\alpha}\int_{-\infty}^{+\infty} \hat{f}_\alpha(\omega)\,|\omega|^{\frac{1}{\alpha}-1}\ \overline{\hat{g}_\alpha(\omega)}d\omega.$$

Thus,

$$< f, g >= \frac{1}{2\pi\alpha} <|\omega|^{\frac{1}{\alpha}-1}\ \hat{f}_\alpha(\omega), \hat{g}_\alpha(\omega)>.$$

**Definition 2.1***: If* $\psi \in L^2(\mathbb{R})$ *,then fractional Wavelet of order* $\alpha(0 < \alpha \le 1)$ *is defined as*

*(2.2)* $$\psi_{\alpha,a,b}(t) = \frac{1}{|a|^{\frac{1}{2\alpha}}}\psi\left(\frac{t-b}{|a|^{\frac{1}{\alpha}}}\right),$$

*where* $a,b\in\mathbb{R}$ *and satisfying the admissibility condition*

(2.3) $$C_{\psi_\alpha}\equiv\int_{-\infty}^{+\infty}\frac{|\widehat{\psi}_\alpha(\omega)|^2}{|\omega|}d\omega<\infty\ .$$

**Definition 2.2:** *If* $\psi\in L^2(\mathbb{R})$, *then the integral transformation*

(2.4) $$W_{\psi_\alpha}[f](a,b)=<f,\psi_{\alpha,a,b}>=\int_{-\infty}^{+\infty}f(t)\overline{\psi_{\alpha,a,b}(t)}dt$$

*is called continuous fractional Wavelet transform of* $f\in L^2(\mathbb{R})$.

**Theorem 2.:** *If* $\psi\in L^2(\mathbb{R})$, *then the following result holds;*

(2.5) $$F_\alpha(\psi_{\alpha,a,b})(\omega)=|a|^{\frac{1}{2\alpha}}\,e^{-i(sign\omega)|\omega|^{\frac{1}{\alpha}}b}\,\widehat{\psi}_\alpha(a\omega)\,.$$

***Proof:*** We take $F_\alpha(\psi_{\alpha,a,b})(\omega)=\int_{-\infty}^{+\infty}e^{-i(sign\omega)|\omega|^{\frac{1}{\alpha}}t}\,\frac{1}{|a|^{\frac{1}{2\alpha}}}\psi\left(\frac{t-b}{|a|^{\frac{1}{\alpha}}}\right)dt$ .

Let $\frac{t-b}{|a|^{\frac{1}{\alpha}}}=u$, then $dt=|a|^{\frac{1}{\alpha}}\,du$,

$$F_\alpha(\psi_{\alpha,a,b})(\omega)=|a|^{\frac{1}{2\alpha}}\int_{-\infty}^{+\infty}e^{-i(sign\omega)|\omega|^{\frac{1}{\alpha}}(|a|^{\frac{1}{\alpha}}u+b)}\psi(u)du$$

$$=|a|^{\frac{1}{2\alpha}}\,e^{-i(sign\omega)|\omega|^{\frac{1}{\alpha}}b}\int_{-\infty}^{+\infty}e^{-i(sign\omega)|a\omega|^{\frac{1}{\alpha}}u}\psi(u)du\,.$$

Thus,

$$F_\alpha(\psi_{\alpha,a,b})(\omega)=|a|^{\frac{1}{2\alpha}}\,e^{-i(sign\omega)|\omega|^{\frac{1}{\alpha}}b}\,\widehat{\psi}_\alpha(a\omega).$$

**Theorem 2.3:** *Let* $\psi\in L^2(\mathbb{R})$, *then for any signal* $f\in L^2(\mathbb{R})$, *the following relation holds;*

(2.6) $$F_\alpha\left(\left(W_{\psi_\alpha}f\right)(a,b)\right)(\omega)=|a|^{\frac{1}{2\alpha}}\,\widehat{f}_\alpha(\omega)\overline{\widehat{\psi}_\alpha(a\omega)}\,.$$

***Proof:*** Continuous fractional Wavelet transform of the signal $f(t)$ is defined as

$$\left(W_{\psi_\alpha} f\right)(a,b) = < f, \psi_{\alpha,a,b} > .$$

Using (2.1), we have

$$\left(W_{\psi_\alpha} f\right)(a,b) = \frac{1}{2\pi\alpha}\left\langle |\omega|^{\frac{1}{\alpha}-1}\,\widehat{f}_\alpha(\omega), \left(F_\alpha\left(\psi_{\alpha,a,b}\right)\right)(\omega)\right\rangle.$$

With the help of (2.5), the above expression becomes

$$\left(W_{\psi_\alpha} f\right)(a,b) = \frac{|a|^{\frac{1}{2\alpha}}}{2\pi\alpha}\int_{-\infty}^{+\infty} e^{i(sign\omega)|\omega|^{\frac{1}{\alpha}}b}\,|\omega|^{\frac{1}{\alpha}-1}\,\widehat{f}_\alpha(\omega)\overline{\widehat{\psi}_\alpha(a\omega)}d\omega$$

$$= F_\alpha^{-1}\left(|a|^{\frac{1}{2\alpha}}\,\widehat{f}_\alpha(\omega)\overline{\widehat{\psi}_\alpha(a\omega)}\right)(b).$$

Thus,

$$F_\alpha\left(\left(W_{\psi_\alpha} f\right)(a,b)\right)(\omega) = |a|^{\frac{1}{2\alpha}}\,\widehat{f}_\alpha(\omega)\overline{\widehat{\psi}_\alpha(a\omega)}.$$

**Theorem 2.4:** *If* $\psi \in L^2(\mathbb{R})$, *then for any functions* $f, g \in L^2(\mathbb{R})$, *we obtain*

*(2.7)*
$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(W_{\psi_\alpha} f\right)(b,a)\overline{\left(W_{\psi_\alpha} g\right)(b,a)}\frac{dbda}{|a|^{\frac{1}{\alpha}+1}} = C_{\psi_\alpha}\left\langle f, g\right\rangle.$$

***Proof:*** We take

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(W_{\psi_\alpha} f\right)(b,a)\overline{\left(W_{\psi_\alpha} g\right)(b,a)}\frac{dbda}{|a|^{\frac{1}{\alpha}+1}}$$

$$= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\left(W_{\psi_\alpha} f\right)(b,a)\overline{\left(W_{\psi_\alpha} g\right)(b,a)}db\right)\frac{da}{|a|^{\frac{1}{\alpha}+1}}.$$

Applying (2.1), we get

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(W_{\psi_\alpha} f\right)(b,a)\overline{\left(W_{\psi_\alpha} g\right)(b,a)}\frac{dbda}{|a|^{\frac{1}{\alpha}+1}}$$

$$= \frac{1}{2\pi\alpha}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}|\omega|^{\frac{1}{\alpha}-1}\left(F_\alpha\left(W_{\psi_\alpha} f\right)(b,a)\right)(\omega)\overline{\left(F_\alpha\left(W_{\psi_\alpha} g\right)(b,a)\right)(\omega)d\omega}\right)\frac{da}{|a|^{\frac{1}{\alpha}+1}}.$$

Using (2.6), we have

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(W_{\psi_\alpha}f\right)(b,a)\overline{\left(W_{\psi_\alpha}g\right)(b,a)}\frac{dbda}{|a|^{\frac{1}{\alpha}+1}}$$

$$=\frac{1}{2\pi\alpha}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}|\omega|^{\frac{1}{\alpha}-1}\,\hat{f}_\alpha(\omega)\overline{\hat{g}_\alpha(\omega)}\,|\hat{\psi}_\alpha(a\omega)|^2\,d\omega\right)\frac{da}{|a|}$$

$$=\frac{1}{2\pi\alpha}\int_{-\infty}^{+\infty}\frac{|\hat{\psi}_\alpha(a\omega)|^2}{|a|}\left(\int_{-\infty}^{+\infty}|\omega|^{\frac{1}{\alpha}-1}\,\hat{f}_\alpha(\omega)\overline{\hat{g}_\alpha(\omega)}d\omega\right)da$$

$$=\frac{C_{\psi_\alpha}}{2\pi\alpha}\left\langle|\omega|^{\frac{1}{\alpha}-1}\,\hat{f}_\alpha(\omega),\hat{g}_\alpha(\omega)\right\rangle.$$

In the view of (2.1), we obtain

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(W_{\psi_\alpha}f\right)(b,a)\overline{\left(W_{\psi_\alpha}g\right)(b,a)}\frac{dbda}{|a|^{\frac{1}{\alpha}+1}}=C_{\psi_\alpha}\left\langle f,g\right\rangle.$$

**Theorem 2.5:** *If* $f\in L^2(\mathbb{R})$, *then* $f$ *can be reconstructed by the formula*

*(2.8)* $$f(t)=\frac{1}{C_{\psi_\alpha}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(W_{\psi_\alpha}f\right)(b,a)\psi_{\alpha,a,b}(t)\frac{dbda}{|a|^{\frac{1}{\alpha}+1}}.$$

***Proof :*** From (2.7) , we have

$$C_{\psi_\alpha}\left\langle f,g\right\rangle=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(W_{\psi_\alpha}f\right)(b,a)\overline{\left(W_{\psi_\alpha}g\right)(b,a)}\frac{dbda}{|a|^{\frac{1}{\alpha}+1}}$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(W_{\psi_\alpha}f\right)(b,a)\overline{\int_{-\infty}^{+\infty}g(t)\overline{\psi_{\alpha,a,b}(t)}dt}\frac{dbda}{|a|^{\frac{1}{\alpha}+1}}$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(W_{\psi_\alpha}f\right)(b,a)\psi_{\alpha,a,b}(t)\frac{dbda}{|a|^{\frac{1}{\alpha}+1}}\overline{g(t)}dt$$

$$= \left\langle \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(W_{\psi_\alpha} f\right)(b,a)\psi_{\alpha,a,b}(t)\frac{dbda}{|a|^{\frac{1}{\alpha}+1}}, g \right\rangle .$$

Since $g$ is any arbitrary element of $L^2(\mathbb{R})$, the inversion formula is given by

$$f(t) = \frac{1}{C_{\psi_\alpha}} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(W_{\psi_\alpha} f\right)(b,a)\psi_{\alpha,a,b}(t)\frac{dbda}{|a|^{\frac{1}{\alpha}+1}} .$$

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