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Conformal *β*-Change of Finsler Metric

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Abstract: The purpose of the present paper is to find the necessary and sufficient conditions under which a conformal β-change of Finsler metric becomes a projective change .We have also found a condition under which a conformal β-change of Finsler metric leads a Douglas space into a Douglas space.

Keywords: FinslerSpace, Finslermetric, conformal β-change, projective change, Douglas space.

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1. Introduction

Let $F^n = \left(M^n, L\right)$ be an n- dimensional Finsler space on the differentiable manifold M^n, equipped with the fundamental function L(x,y). B. N. Prasad and Bindu Kumari[1] and C. Shibata[2] have considered the β- change of Finsler metric given by

$$L^*(x,y) = f(L,\beta),$$

where f is positively homogeneous function of degree one in L and β, where β given by

$$\beta(x,y) = b_i(x) y^i$$

is a one-form on M^n .

The conformal theory of Finsler space was initiated by M. S. Knebelman[3] in 1929 and has been investigated in detail by many authors (Hashiguchi[4], Izumi[5,6] and Kitayama[7]). The conformal change is defined as

$$L^{*}(x,y)\to e^{\sigma(x)}L(x,y),$$

where $\sigma(x)$ is a function of position only and known as conformal factor.

In this paper we have combined the above two changes and have introduced another Finsler metric defined as

(1.1) $$\overline{L}(x,y)=e^{\sigma}f(L,\beta),$$

where $\sigma(x)$ is a function of x and $\beta(x,y)=b_i(x)y^i$ is a 1-form on M^n.

This conformal change of $(L,\beta)-$ metric will be called as conformal $\beta-$ change of Finsler metric. When $\sigma=0$, it reduces to a $\beta-$change. When $\sigma=\text{constant}$, it becomes a homothetic $\beta-$change. When $f(L,\beta)$ has special forms as $L+\beta$, $\frac{L^2}{L-\beta}$, $\frac{L^2}{\beta}$, $\frac{L^{m+1}}{\beta^m}$ $(m\neq 0,-1)$, we get conformal Randers change, conformal Matsumoto change, conformal Kropina change, conformal generalized Kropina change of Finsler metric respectively. The Finsler space equipped with the metric $\overline{L}$ given by (1.1) will be denoted by $\bar{F}^n$. Throughout the paper the quantities corresponding to $\bar{F}^n$ will be denoted by putting bar on the top of them. The fundamental quantities of F^n are given by

$$g_{ij}=\frac{1}{2}\frac{\partial^2 L^2}{\partial y^i\partial y^j},\quad l_i=\frac{\partial L}{\partial y^i}\quad\text{and}\quad h_{ij}=L\frac{\partial^2 L^2}{\partial y^i\partial y^j}=g_{ij}-l_i l_j\ .$$

We shall denote the partial derivatives with respect to x^i and y^i by ∂_i and $\dot{\partial}_i$ respectively and write

$$L_i=\dot{\partial}_i L,\ L_{ij}=\dot{\partial}_j\dot{\partial}_i L,\ L_{ijk}=\dot{\partial}_k\dot{\partial}_j\dot{\partial}_i L\,.$$

Then $L_i=l_i$, $L^{-1}h_{ij}=L_{ij}$. The geodesics of F^n are given by the system of differential equations

$$\frac{d^2x^i}{ds^2}+2G^i\left(x,\frac{dx}{ds}\right)=0,$$

where $G^i(x,y)$ are positively homogeneous of degree two in y^i and are given by

$$2G^i=g^{ij}\left(y^r\dot{\partial}_j\partial_r F-\partial_j F\right),\ F=\frac{L^2}{2},$$

where g^{ij} are the inverse of g_{ij}.

Berwald connection $B\Gamma=\left(G^i_{jk},G^i_j,0\right)$ of Finsler space is given by[8]:

$$G^i_j=\frac{\partial G^i}{\partial y^j},\ G^i_{jk}=\frac{\partial G^i_j}{\partial y^k}.$$

The Cartan's connection $\left(F^i_{jk},G^i_j,C^i_{jk}\right)$ is constructed from the metric function$L$ with the help of following axioms[8]:

(1) Cartan's connection CΓ is v-metrical.
(2) Cartan's connection CΓ is h-metrical.
(3) The $(v)v$-torsion tensor field S of Cartan's connection vanishes.
(4) The $(h)h$-torsion tensor field T of Cartan's connection vanishes.
(5) The deflection tensor field D of Cartan's connection vanishes.

The h- and v- covariant derivatives with respect to Cartan's connection aredenoted by $_{|k}$ and $|_k$ respectively. It is clear that the h-covariant derivative of L with respect to BΓ and CΓ is the same and vanishes identically. Further-more, the h-covariant derivatives of L_i, L_{ij} with respect to CΓ are also zero. We shall write

$$2r_{ij}=b_{i|j}+b_{j|i},\quad 2s_{ij}=b_{i|j}-b_{j|i}.$$

2. Difference Tensor of Conformal *β*-Change

The conformal β- change of Finsler metric L is given by

$$\overline{L}(x,y)=e^{\sigma}f(L,\beta),$$

where$f$ is positively homogeneous function of degree one in L and β. Homogeneity of $f$gives

$$Lf_1+\beta f_2=f,$$

where subscripts "1" and "2" denote the partial derivatives with respect to L and βrespectively.

Differentiating above equations with respect to L and β respectively, we get

$$Lf_{12}+\beta f_{22}=0 \text{ and } Lf_{11}+\beta f_{21}=0.$$

Hence, we have

$$\frac{f_{11}}{\beta^2}=\frac{-f_{12}}{L\beta}=\frac{f_{22}}{L^2},$$

which gives

$$f_{11}=\beta^2\omega,\quad f_{12}=-L\beta\omega,\quad f_{22}=L^2\omega,$$

where Weierstrass function ω is positively homogeneous of degree-3 in L and β . Therefore

$$L\omega_1+\beta\omega_2+3\omega=0,$$

where ω_1 and ω_2 are positively homogeneous of degree – 4 in L and β .

Throughout the paper we frequently use the above equations without quoting them. Also we have assumed that f is not a linear function of L and β so that $\omega \neq 0$.We now put

(2.1) $$\bar{G}^i=G^i+D^i .$$

Then $\bar{G}_j^i=G_j^i+D_j^i$ and $\bar{G}^i_{jk}=G^i_{jk}+D^i_{jk}$, where $D_j^i=\dot{\partial}_j D^i$ and $D^i_{jk}=\dot{\partial}_k D_j^i$.

The tensors D^i, D_j^i and D_{jk}^i are positively homogeneous in y^i of degree two, one and zero respectively. To find D^i we deal with equation $L_{ij\,|k}=0$[9], i.e.,

(2.2) $$\partial_k L_{ij}-L_{ijr}G_k^r-L_{rj}F^r_{ik}-L_{ir}F^r_{jk}=0 .$$

Since $\dot{\partial}_i\beta=b_i$,from (1.1),we have

(2.3) $$\bar{L}_i=e^{\sigma}\left(f_1L_i+f_2b_i\right),$$

$$\bar{L}_{ij}=e^{\sigma}\left[f_1L_{ij}+\beta^2\omega L_iL_j-L\beta\omega\left(L_ib_j+L_jb_i\right)+L^2b_ib_j\right],$$

$$\bar{L}_{ijk}=e^{\sigma}\begin{bmatrix} f_1L_{ijk}+\beta^2\omega\left(L_iL_{jk}+L_jL_{ik}+L_kL_{ij}\right)-L\beta\omega\left(b_iL_{jk}+b_jL_{ik}+b_kL_{ij}\right) \\ +\beta\left(2\omega+\beta\omega_2\right)\left(L_iL_jb_k+L_iL_kb_j+L_jL_kb_i\right)+\beta^2\omega_1L_iL_jL_k \\ -L\left(\omega+\beta\omega_2\right)\left(b_ib_jL_k+b_ib_kL_j+b_jb_kL_i\right)+L^2\omega_2b_ib_jb_k \end{bmatrix},$$

$$\partial_j\bar{L}_i=e^{\sigma}\begin{bmatrix} f_1\partial_jL_i+\omega\left(\beta^2L_i-L\beta b_i\right)\partial_jL+\omega\left(L^2b_i-L\beta L_i\right)\partial_j\beta \\ +f_2\partial_jb_i+\left(f_1L_i+f_2b_i\right)\sigma_j \end{bmatrix},$$

$$\partial_k\bar{L}_{ij}=e^{\sigma}\begin{bmatrix} f_1\partial_kL_{ij}+\begin{Bmatrix}\beta^2\omega L_{ij}-\beta\left(\omega+L\omega_1\right)\left(L_ib_j+L_jb_i\right) \\ +\beta^2\omega_1L_iL_j+L\left(2\omega+L\omega_1\right)b_ib_j\end{Bmatrix}\partial_kL \\ +\begin{Bmatrix}-L\beta\omega L_{ij}+\beta\left(2\omega+\beta\omega_2\right)L_iL_j \\ -L\left(\omega_1+\beta\omega_2\right)\left(L_ib_j+L_jb_i\right)+L^2\omega_2b_jb_i\end{Bmatrix}\partial_k\beta \\ +\left(\beta^2\omega L_j-L\beta\omega b_j\right)\partial_kL_i+\left(\beta^2\omega L_i-L\beta\omega b_i\right)\partial_kL_j \\ +\omega\left(L^2b_j-L\beta L_j\right)\partial_kb_i+\omega\left(L^2b_i-L\beta L_i\right)\partial_kb_j \\ +\left\{f_1L_{ij}+\beta^2\omega L_iL_j-L\beta\omega\left(L_ib_j+L_jb_i\right)+L^2\omega b_jb_i\right\}\sigma_k \end{bmatrix}$$

where $\sigma_k=\dfrac{\partial\sigma}{\partial x^k}$.

Since $\bar{L}_{ij|k}=0$ in $\bar{F}^n$, after using (2.1), we have

$$\partial_k\bar{L}_{ij}-\bar{L}_{ijr}\bar{G}^r_k-\bar{L}_{rj}\bar{F}^r_{ik}-\bar{L}_{ir}\bar{F}^r_{jk}=0. \tag{2.4}$$

Substituting in the above equation the values of $\partial_k\bar{L}_{ij}$, $\bar{L}_{ir}$ and $\bar{L}_{ijk}$ from (2.3) in (2.4) and then contracting the equation thus obtained with y^k, we get

$$\left\{\begin{array}{l} 2\bar{L}_{ijr}D^r+\bar{L}_{jr}D^r_i+\bar{L}_{ir}D^r_j-\omega\left(L^2b_j-L\beta L_j\right)\left(r_{i0}+s_{i0}\right) \\ -\omega\left(L^2b_i-L\beta L_i\right)\left(r_{j0}+s_{j0}\right)-\begin{Bmatrix}-L\beta\omega L_{ij}+\beta\left(2\omega+\beta\omega_2\right)L_iL_j \\ -L\left(\omega+\beta\omega_2\right)\left(L_ib_j+L_jb_i\right)+L^2\omega_2b_jb_i\end{Bmatrix}r_{00} \\ +\left\{f_1L_{ij}+\beta^2\omega L_iL_j-L\beta\omega\left(L_ib_j+L_jb_i\right)+L^2\omega b_jb_i\right\}\sigma_0=0, \end{array}\right. \tag{2.5}$$

where $'0'$ stands for contraction with y^k,viz., $r_{j0}=r_{jk}y^k$, $r_{00}=r_{jk}y^jy^k$, $\sigma_0=\sigma_i y^i$ and we have used the fact that $D^i{}_{jk}y^k={}^cD^i{}_{jk}y^k=D^i{}_j$ **9**, where ${}^cD^i{}_{jk}=\bar{F}^i{}_{jk}-F^i{}_{jk}$.

Next, we deal with $\bar{L}_{i|j}=0$, that is,

$$\partial_j\bar{L}_i-\bar{L}_{ir}\bar{G}^r_j-\bar{L}_r\bar{F}^r_{ij}=0. \tag{2.6}$$

Putting the values of $\partial_j\bar{L}_i$, $\bar{L}_{ir}$ and $\bar{L}_r$ from (2.3) in (2.6) we get,

$$\begin{aligned} f_2b_{i|j}&=\left\{f_1L_{ir}+\beta^2\omega L_iL_r-L\beta\omega\left(L_ib_r+L_rb_i\right)+L^2\omega b_ib_r\right\}D^r_j\\ &\quad+\left(f_1L_r+f_2b_r\right){}^cD^r_{ij}-\left(L^2\omega b_i-L\beta\omega L_i\right)\left(r_{0j}+s_{0j}\right)-\left(f_1L_i+f_2b_i\right)\sigma_j, \end{aligned}$$

hence after using $2r_{ij}=b_{i|j}+b_{j|i}$ and $2s_{ij}=b_{i|j}-b_{j|i}$,we get

$$\left\{\begin{aligned} 2f_2r_{ij}&=\left\{f_1L_{ir}+\beta^2\omega L_iL_r-L\beta\omega\left(L_ib_r+L_rb_i\right)+L^2\omega b_ib_r\right\}D^r_j\\ &\quad+\left\{f_1L_{jr}+\beta^2\omega L_jL_r-L\beta\omega\left(L_jb_r+L_rb_j\right)+L^2\omega b_jb_r\right\}D^r_i\\ &\quad-\left(L^2\omega b_i-L\beta\omega L_i\right)\left(r_{0j}+s_{0j}\right)-\left(L^2\omega b_j-L\beta\omega L_j\right)\left(r_{0i}+s_{0i}\right)\\ &\quad-\left(f_1L_i+f_2b_i\right)\sigma_j-\left(f_1L_j+f_2b_j\right)\sigma_i+2\left(f_1L_r+f_2b_r\right){}^cD^r_{ij}, \end{aligned}\right. \tag{2.7}$$

$$\left\{\begin{aligned} 2f_2s_{ij}&=\left\{f_1L_{ir}+\beta^2\omega L_iL_r-L\beta\omega\left(L_ib_r+L_rb_i\right)+L^2\omega b_ib_r\right\}D^r_j\\ &\quad+\left\{f_1L_{jr}+\beta^2\omega L_jL_r-L\beta\omega\left(L_jb_r+L_rb_j\right)+L^2\omega b_jb_r\right\}D^r_i\\ &\quad-\left(L^2\omega b_i-L\beta\omega L_i\right)\left(r_{0j}+s_{0j}\right)+\left(L^2\omega b_j-L\beta\omega L_j\right)\left(r_{0i}+s_{0i}\right)\\ &\quad-\left(f_1L_i+f_2b_i\right)\sigma_j+\left(f_1L_j+f_2b_j\right)\sigma_i+2\left(f_1L_r+f_2b_r\right){}^cD^r_{ij}. \end{aligned}\right. \tag{2.8}$$

Subtracting (2.7) from (2.5) and contracting the resulting equation with y^i, we get

$$\left\{\begin{aligned} &-2\left\{f_1L_{jr}+\beta^2\omega L_jL_r-L\beta\omega\left(L_jb_r+L_rb_j\right)+L^2\omega b_jb_r\right\}D^r\\ &+\left(L^2\omega b_j-L\beta\omega L_j\right)r_{00}+2f_2r_{0j}=2\bar{L}_rD^r_j-\left(f_1L+f_2\beta\right)\sigma_j-\left(f_1L_j+f_2b_j\right)\sigma_0. \end{aligned}\right. \tag{2.9}$$

Contracting (2.9) with y^j , we get

(2.10) $$\{f_1L_r+f_2b_r\}D^r=\frac{1}{2}(f_2r_{00}+f\sigma_0).$$

Subtracting (2.8) from (2.5) and contracting the resulting equation with y^j, we get

(2.11) $$\begin{cases}\{f_1L_{ir}+\beta^2\omega L_iL_r-L\beta\omega(L_ib_r+L_rb_i)+L^2\omega b_ib_r\}D^r\\ =f_2s_{i0}+\frac{1}{2}(L^2\omega b_i-L\beta\omega L_i)r_{00}+L\beta\omega(L_i\beta-Lb_i)y^k\sigma_k\\ \qquad+\frac{1}{2}(f_1L_i+f_2b_i)\sigma_0-\frac{1}{2}f\sigma_i.\end{cases}$$

In view of $LL_{ir}=g_{ir}-L_iL_r$, equation (2.11) can be written as

(2.12) $$\begin{cases}\frac{f_1}{L}g_{ir}D^r+\left\{\left(-\frac{f_1}{L}+\beta^2\omega\right)L_i-L\beta\omega b_i\right\}L_rD^r+\{L^2\omega b_i-L\beta\omega L_i\}b_rD^r\\ \quad=f_2s_{i0}+\frac{1}{2}(L^2\omega b_i-L\beta\omega L_i)r_{00}+\frac{1}{2}(f_1L_i+f_2b_i)\sigma_0-\frac{1}{2}f\sigma_i.\end{cases}$$

Contracting (2.12) by$b^i=g^{ij}b_j$, we get

(2.13) $$\begin{cases}\left\{-\frac{f_1\beta}{L^2}-L\beta\omega\Delta\right\}L_rD^r+\left\{\frac{f_1}{L}+L^2\omega\Delta\right\}b_rD^r\\ \quad=\frac{L^2\omega\Delta}{2}r_{00}+f_2s_0+\frac{1}{2}\left(\frac{f_1\beta}{L}+f_2b^2\right)-\frac{1}{2}f\sigma_1,\end{cases}$$

where$\Delta=b^2-\frac{\beta^2}{L^2}$ and $\sigma_1=\sigma_ib^i$.

The equation (2.10) and (2.13) are algebraic equations in $L_rD^r$and b_rD^r , whose solution is given by

(2.14) $$b_rD^r=\frac{\left(f_1f_2\beta+L^3\omega f\Delta\right)r_{00}+2f_1f_2L^2s_0+\left\{\begin{matrix}\beta\left(f_1+L^3\omega\Delta\right)\\ +L\left(f_1{}^2\beta+Lb^2f_1f_2\right)\end{matrix}\right\}\sigma_0-ff_1L^2\sigma_1}{2f\left(f_1+L^3\omega\Delta\right)}$$

and

(2.15) $$L_r D^r = \frac{Lf_1 f_2 r_{00} - f_2^2 L^2 s_0 + L\left\{ f\left(f_1 + L^3\omega\Delta \right) - \left(Lf_2^2 b^2 + \beta f_1 f_2 \right) \right\}\sigma_0 - ff_2 L^2 \sigma_1}{2f\left(f_1 + L^3\omega\Delta \right)}.$$

Contracting (2.12) by g^{ij} and putting the values of $L_r D^r$ and $b_r D^r$,we get

(2.16) $$\begin{cases} D^i = \left\{ \dfrac{\left(f_1 f_2 - L\beta\omega f \right)\left(f_1 r_{00} - 2Lf_2 s_0 \right)}{2ff_1\left(f_1 + L^3\omega\Delta \right)} + \sigma_0 + \dfrac{\left(f_1 f_2 - L\beta\omega f \right)\begin{bmatrix} Lf\sigma_1 \\ -\left(Lf_2 b^2 + \beta f_1 \right)\sigma_0 \end{bmatrix}}{2ff_1\left(f_1 + L^3\omega\Delta \right)} \right\} y^i \\ + \left\{ \dfrac{L^3\omega\left(f_1 r_{00} - 2Lf_2 s_0 \right)}{2f_1\left(f_1 + L^3\omega\Delta \right)} + \dfrac{Lf_2}{2f_1}\sigma_0 + \dfrac{L^3\omega\left[Lf\sigma_1 - \left(Lf_2 b^2 + \beta f_1 \right)\sigma_0 \right]}{2f_1\left(f_1 + L^3\omega\Delta \right)} \right\} b^i \\ - \dfrac{Lf}{2f_1}\sigma_j g^{ij} + \dfrac{Lf_2}{f_1} s^i{}_0 , \end{cases}$$

where $l^i = y^i L^{-1}$.

Proposition 2.1: *The difference tensor* $D^i = \bar{G}^i - G^i$ *of conformal* β*-change of Finsler metric is given by (2.16).*

3. Projective Change of FinslerMetric

The Finsler space $\bar{F}^n$ is said to be projective to Finsler space F^n if every geodesic of F^n is transformed to a geodesic of $\bar{F}^n$ and vice-versa. It is well known that the change $L \to \bar{L}$ is projective iff $\bar{G}^i = G^i + P(x,y)y^i$, where $P(x,y)$ is a homogeneous scalar function of degree one in y^i, called projective factor[10]. Thus from (2.1) it follow that $L \to \bar{L}$ is projective iff $D^i = Py^i$. Now we consider that the changes $L \to \bar{L}$ is projective .Then from equation (2.16), we have

$$(3.1)\begin{cases} Py^i=\left\{\dfrac{(f_1f_2-L\beta\omega f)(f_1r_{00}-2Lf_2s_0)}{2ff_1(f_1+L^3\omega\Delta)}+\sigma_0+\dfrac{(f_1f_2-L\beta\omega f)\begin{bmatrix}Lf\sigma_1\\ -(Lf_2b^2+\beta f_1)\sigma_0\end{bmatrix}}{2ff_1(f_1+L^3\omega\Delta)}\right\}y^i \\ +\left\{\dfrac{L^3\omega(f_1r_{00}-2Lf_2s_0)}{2f_1(f_1+L^3\omega\Delta)}+\dfrac{Lf_2}{2f_1}\sigma_0+\dfrac{L^3\omega\left[Lf\sigma_1-(Lf_2b^2+\beta f_1)\sigma_0\right]}{2f_1(f_1+L^3\omega\Delta)}\right\}b^i \\ -\dfrac{Lf}{2f_1}\sigma_jg^{ij}+\dfrac{Lf_2}{f_1}s^i{}_0, \end{cases}$$

Contracting (3.1) with $y_i(=g_{ij}y^j)$ and using the fact that $s_0^iy_i=0$ and $y_iy^i=L^2$, we get

$$(3.2)\qquad P=\frac{f_2(f_1r_{00}-2Lf_2s_0)}{2f(f_1+L^3\omega\Delta)}+\frac{f_2Lf\sigma_1+\left\{f(f_1+L^3\omega\Delta)-f_2(Lf_2b^2+\beta f_1)\sigma_0\right\}}{2f(f_1+L^3\omega\Delta)}.$$

Putting the value of P from (3.2) in (3.1) ,we get

$$(3.3)\qquad \begin{cases}(L\beta\omega y^i-L^3\omega b^i)(f_1r_{00}-2Lf_2s_0)-(f_1y^i+f_2b^i)(f_1+L^3\omega\Delta)\\ +\left\{Lf\sigma_1-(Lf_2b^2+\beta f_1)\sigma_0\right\}(L\beta\omega y^i-L^3\omega b^i)\\ \quad=-Lf(f_1+L^3\omega\Delta)\sigma_jg^{ij}+2Lf_2(f_1+L^3\omega\Delta)s^i{}_0\,.\end{cases}$$

Transvecting (3.3) by b_i, we get

$$(3.4)\qquad r_{00}=\frac{-2Lf_2s_0+Lf\sigma_1-(Lf_2b^2+\beta f_1)\sigma_0}{L^3\omega\Delta}.$$

Substituting the value of r_{00} from (3.4) in (3.2), we get

$$(3.5)\qquad P=\frac{-2f_2^2s_0+Lff_2\sigma_1-f_2(Lf_2b^2+\beta f_1)\sigma_0+fL^3\omega\Delta\sigma_0}{2fL^3\omega\Delta}$$

Substituting the value of r_{00} from (3.4) in (3.3), we get

(3.6) $$s^i_0=\left(b^i-\frac{\beta}{L^2}y^i\right)\frac{s_0}{\Delta}+\frac{f}{2f_2}\sigma_j g^{ij}-\frac{\left(f_1 y^i+f_2 b^i\right)\sigma_0}{2Lf_2}+\frac{\left\{Lf\sigma_1-\left(Lf_2 b^2+\beta f_1\right)\sigma_0\right\}}{2f_2 L^4\omega\Delta}.$$

The equations (3.4) and (3.6) give the necessary conditions under which the change $L\to\bar{L}$ becomes a projective change.

Conversely, if conditions (3.4) and (3.6) are satisfied, then putting the values of r_{00} and s_0^i from (3.4) and (3.6) respectively in (2.16), we get

$$D^i=\frac{-2f_2^2 s_0+Lff_2\sigma_1-f_2\left(Lf_2 b^2+\beta f_1\right)\sigma_0+fL^3\omega\Delta\sigma_0}{2fL^3\omega\Delta}y^i$$

i.e. $D^i=Py^i$, where P is given by (3.5). Thus $\overline{F}^n$ is projective to F^n .

Theorem 3.1: *The conformal β-change of Finsler metric is projective iff (3.4) and (3.6) hold good, the projective factor P is given by (3.5).*

When $\sigma=0$, the change (1.1) is simply a β-change of original metric and the condition (3.4) reduces to

(3.7) $$r_{00}=\frac{-2Lf_2 s_0}{L^3\omega\Delta}.$$

where as the condition (3.6) reduces to

(3.8) $$s^i_0=\left(b^i-\frac{\beta}{L^2}y^i\right)\frac{s_0}{\Delta}.$$

Thus we get

Corollary 3.1: *The β-change of Finsler metric is projective iff (3.7) and (3.8) hold good .*

This result has been investigated in[12].

4. Douglas Space

The Finsler space F^n is called a Douglas space iff $G^i y^j - G^j y^i$ is homogeneous polynomial of degree three in y^i [11].We shall write *hp(r)* to denote a homogeneous polynomial in y^i of degree *r*. If we write $B^{ij} = D^i y^j - D^j y^i$, then from (2.16), we get

$$(4.1)\begin{cases} B^{ij} = \left\{ \dfrac{L^3\omega\left(f_1 r_{00} - 2Lf_2 s_0\right)}{2f_1\left(f_1 + L^3\omega\Delta\right)} + \dfrac{Lf_2}{2f_1}\sigma_0 + \dfrac{L^3\omega\left[Lf\sigma_1 - \left(Lf_2 b^2 + \beta f_1\right)\sigma_0\right]}{2f_1\left(f_1 + L^3\omega\Delta\right)} \right\}\left(b^i y^j - b^j y^i\right) \\ \quad + \dfrac{Lf_2}{f_1}\left(s^i{}_0 y^j - s^j{}_0 y^i\right). \end{cases}$$

If a Douglas space is transformed to a Douglas space by a conformal β-change of Finsler metric (2.1) then B^{ij} must be hp (3) and vice-versa.

Theorem 4.1: *The conformal β-change of Finsler metric leads a Douglas space into a Douglas space iff B^{ij} given by (4.1) is hp(3).*

When $\sigma = 0$, the change (1.1)is simply a β-change of original metric and the condition (4.1) reduces to

$$(4.2) \quad B^{ij} = \left\{ \frac{L^3\omega\left(f_1 r_{00} - 2Lf_2 s_0\right)}{2f_1\left(f_1 + L^3\omega\Delta\right)} \right\}\left(b^i y^j - b^j y^i\right) + \frac{Lf_2}{f_1}\left(s^i{}_0 y^j - s^j{}_0 y^i\right).$$

Thus we get

Corollary 4.1:*The β-change of Finsler metric leads a Douglas space into a Douglas space iff B^{ij} given by (4.2) is hp(3).*

This result has been investigated in[12].

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