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# Certain Results on 3-Dissection of Ratio of Infinite Products

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**Abstract**: The m-dissection of the power series $P=\sum_{n=0}^{\infty} a_n q^n$ is the representation of $P$ as $P=P_0+P_1+\ldots\ldots\ldots.+P_{m-1}$, where $P_k=\sum_{n=0}^{\infty} a_{mn+k} q^{mn+k}$. The m-dissection of continued fractions are represented in terms of power series and infinite products. In this paper, an attempt has been made to obtain results on 3- dissection of ratio of infinite products.

## 1. Introduction

The oldest and the most famous theorem associated with Ramanujan's career is the Rogers Ramanujan continued fraction given by

$$C(q)=1+\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\ldots\ldots\ldots\ldots\ldots\ldots=\frac{(q^2,q^3;q^5)_\infty}{(q^2,q^3;q^5)_\infty}.$$

Andrew[1] and Hirschhorn[2] have given the 2-dissection and 5-dissection of $C(q)$

$$R(q)=1+\frac{q(1+q)}{1+}\frac{q^2(1+q^2)}{1+}\frac{q^3(1+q^3)}{1+}\ldots\ldots\ldots\ldots\ldots\ldots$$

Ramanujan[3] gave the following result for $R(q)$

$$(1.1)\qquad R(q)=\frac{(q^3,q^3;q^6)_\infty}{(q,q^5;q^6)_\infty}$$

Neera A. Bhagirathi[4] obtained above result as special case.

The m-dissection of the power series $P=\sum_{n=0}^{\infty} a_n q^n$ is the representation of P

as $$P = P_0 + P_1 + \ldots\ldots\ldots + P_{m-1}, \text{ where } P_k = \sum_{n=0}^{\infty} a_{mn+k} q^{mn+k}$$

Neera A. Herbert[5] has obtained 4-dissection of Ramanujan's continued fraction .

Here an attempt has been made to obtain 3-dissection of ratio of infinite products.

## 2. Notations

Suppose that $|q|<1$, where $q$ is non-zero complex number, this condition ensures that all the infinite products that we use will converge. We will use the notation,

(2.1) $$(z;q)_\infty = \prod_{n=0}^{\infty}\left(1-zq^n\right),$$

(2.2) $$[z;q]_\infty = (z;q)_\infty \left(z^{-1}q;q\right)_\infty \qquad \textit{for } z \neq 0$$

and often we write

(2.3) $$[z_1, z_2, \ldots\ldots\ldots\ldots\ldots z_n; q]_\infty = [z_1;q]_\infty [z_2;q]_\infty \ldots\ldots\ldots\ldots [z_n;q]_\infty$$

The following facts can be easily verified;

(2.4) $$\left[z^{-1};q\right]_\infty = -z^{-1}[z;q]_\infty = [zq;q]_\infty,$$

(2.5) $$\left[z, zq; q^2\right]_\infty = [z;q]_\infty,$$

(2.6) $$[z,-z;q]_\infty = \left[z^2;q^2\right]_\infty,$$

(2.7) $$\left[z^{-1}q;q\right]_\infty = [z\ \ ;q\ \ ]_\infty,$$

(2.8) $$[-1;q]_\infty \left[q;q^2\right]_\infty = 2.$$

Also, we have the following general relations;

Suppose
$a_1, a_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, a_n; b_1, b_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots, b_n \in C \setminus \{0\}$ satisfy

i) $a_i \neq q^n a_j$ for $i \neq j$ and any $n \in \mathbb{Z}$

ii) $a_1 a_2 \ldots\ldots\ldots a_n = b_1 b_2 \ldots\ldots\ldots\ldots b_n$ .

Then,

(2.9)
$$\sum_{i=1}^{n} \frac{\prod_{j=1}^{n} \left[a_i^{-1} b_j; q\right]_\infty}{\prod_{j=1, j\neq i}^{n} \left[a_i^{-1} a_j; q\right]_\infty} = 0.$$

This theorem appears without proof as given by Slater[6] and with a proof as given by Lewis[7].

## 3. Main Results

(3.1)
$$\begin{aligned}\frac{\left[q^{3j}, q^{9j}; q^{12j}\right]_\infty}{\left[q^{j}, q^{7j}; q^{12j}\right]_\infty} &= \frac{\left[q^{2j}, q^{4j}; q^{12j}\right]_\infty \left[q^{6j}; q^{24j}\right]_\infty}{\left[q^{6j}; q^{12j}\right]_\infty \left[q^{2j}, q^{4j}, q^{12j}; q^{24j}\right]_\infty} \\ &+ \frac{q^{2j}\left[q^{2j}, q^{4j}; q^{12j}\right]_\infty \left[q^{6j}; q^{24j}\right]_\infty}{\left[q^{6j}; q^{12j}\right]_\infty \left[q^{4j}, q^{12j}, q^{14j}; q^{24j}\right]_\infty} \\ &+ \frac{q^{j}\left[q^{2j}, q^{2j}, -q^{10j}, -q^{6j}; q^{12j}\right]_\infty \left[q^{2j}; q^{24j}\right]_\infty}{\left[-q^{j}, q^{j}, -q^{2j}; q^{12j}\right]_\infty \left[q^{12j}; q^{24j}\right]_\infty},\end{aligned}$$

(3.2)
$$\begin{aligned}\frac{[q^{j}, q^{7j}; q^{12j}]_\infty}{[q^{3j}, q^{9j}; q^{12j}]_\infty} &= \frac{\left[q^{2j}, q^{4j}; q^{12j}\right]_\infty \left[q^{2j}, q^{14j}; q^{24j}\right]_\infty}{\left[q^{6j}; q^{12j}\right]_\infty \left[q^{2j}, q^{4j}, q^{6j}, q^{12j}; q^{24j}\right]_\infty} \\ &+ \frac{q^{2j}\left[q^{2j}, q^{4j}; q^{12j}\right]_\infty \left[q^{2j}, q^{14j}; q^{24j}\right]_\infty}{\left[q^{6j}; q^{12j}\right]_\infty \left[q^{4j}, q^{6j}, q^{12j}, q^{14j}; q^{24j}\right]_\infty} \\ &- \frac{q^{j}\left[q^{2j}, q^{2j}, -q^{10j}, -q^{6j}; q^{12j}\right]_\infty \left[q^{2j}, q^{2j}, q^{14j}; q^{24j}\right]_\infty}{\left[-q^{j}, q^{j}, -q^{2j}; q^{12j}\right]_\infty \left[q^{6j}, q^{12j}, q^{18j}; q^{24j}\right]_\infty},\end{aligned}$$

where $|q^j| < 1$ and j is any positive integer.

**Proof**: To prove (3.1), we consider

$$\frac{\left[q^{3j},q^{9j};q^{12j}\right]_\infty}{\left[q^{j},q^{7j};q^{12j}\right]_\infty}=R(q^j), \tag{3.3}$$

where $|q^j|<1$ and j is any positive integer.

Now, setting $(a_1,a_2,a_3,a_4;b_1,b_2,b_3,b_4)=(1,-1,q^j,q^{7j};q^{3j},q^{9j},-q^{-2j},q^{-2j})$ and taking $q^{12j}$ for $q$ in (2.9), we get

$$\frac{\left[q^{3j},q^{9j},-q^{-2j},q^{-2j};q^{12j}\right]_\infty}{\left[-1,q^{j},q^{7j};q^{12j}\right]_\infty}+\frac{\left[-q^{3j},-q^{9j},q^{-2j},-q^{-2j};q^{12j}\right]_\infty}{\left[-1,-q^{j},-q^{7j};q^{12j}\right]_\infty}$$
$$+\frac{\left[q^{2j},q^{8j},-q^{-3j},q^{-3j};q^{12j}\right]_\infty}{\left[q^{-j},-q^{-j},q^{6j};q^{12j}\right]_\infty}+\frac{\left[q^{-4j},q^{2j},-q^{-9j},q^{-9j};q^{12j}\right]_\infty}{\left[q^{-7j},-q^{-7j},q^{-6j};q^{12j}\right]_\infty}=0.$$

By applying (2.4) to (2.8) in above equation, we get

$$\frac{\left[q^{3j},q^{9j};q^{12j}\right]_\infty}{\left[q^{j},q^{7j};q^{12j}\right]_\infty}+\frac{\left[-q^{3j},-q^{9j};q^{12j}\right]_\infty}{\left[-q^{j},-q^{7j};q^{12j}\right]_\infty}=\frac{2}{\left[q^{12j};q^{24j}\right]_\infty\left[q^{4j};q^{24j}\right]\left(-q^{-4j}\right)}$$
$$\times\left[\frac{-q^{-4j}\left[q^{2j},q^{8j};q^{12j}\right]_\infty}{\left[q^{6j};q^{12j}\right]_\infty}\frac{\left[q^{6j};q^{24j}\right]_\infty}{\left[q^{2j};q^{24j}\right]_\infty}-\frac{q^{-2j}\left[q^{4j},q^{2j};q^{12j}\right]_\infty}{\left[q^{6j};q^{12j}\right]_\infty}\frac{\left[q^{18j};q^{24j}\right]_\infty}{\left[q^{14j};q^{24j}\right]_\infty}\right]$$

By using (2.2) we get

$$R\left(q^j\right)+R\left(-q^j\right)=\frac{2\left[q^{2j},q^{4j};q^{12j}\right]_\infty\left[q^{6j};q^{24j}\right]_\infty}{\left[q^{6j};q^{12j}\right]_\infty\left[q^{4j},q^{12j};q^{24j}\right]_\infty}\times\left[\frac{1}{\left[q^{2j};q^{24j}\right]_\infty}+\frac{q^{2j}}{\left[q^{14j};q^{24j}\right]_\infty}\right]. \tag{3.4}$$

Let $\alpha_1\left(q^{2j}\right)=\frac{1}{2}\left[R\left(q^j\right)+R\left(-q^j\right)\right],$

(3.5) $$\alpha_1(q^{2i})=\frac{\left[q^{2j},q^{4j};q^{12j}\right]_\infty\left[q^{6j};q^{24j}\right]_\infty}{\left[q^{6j};q^{12j}\right]_\infty\left[q^{2j},q^{4j},q^{12j};q^{24j}\right]_\infty}+\frac{q^{2j}\left[q^{2j},q^{4j};q^{12j}\right]_\infty\left[q^{6j};q^{24j}\right]_\infty}{\left[q^{6j};q^{12j}\right]_\infty\left[q^{4j},q^{12j},q^{14j};q^{24j}\right]_\infty}.$$

Again setting $\left(a_1,a_2,a_3,a_4;b_1,b_2,b_3,b_4\right)=\left(1,-1,q^{11j},-q^{13j};q^{9j},q^{9j},-q^{j},-q^{5j};q^{12j}\right)$ and taking $q^{12j}$ for $q$ in (2.9), we get

$$\frac{\left[q^{9j},q^{9j},-q^{j},-q^{5j};q^{12j}\right]_\infty}{\left[-1,q^{11j},-q^{13j};q^{12j}\right]_\infty}+\frac{\left[-q^{9j},-q^{9j},q^{j},q^{5j};q^{12j}\right]_\infty}{\left[-1,-q^{11j},q^{13j};q^{12j}\right]_\infty}+$$

$$\frac{\left[q^{-2j},q^{-2j},-q^{-10j},-q^{-6j};q^{12j}\right]_\infty}{\left[q^{-11j},-q^{-11j},-q^{2j};q^{12j}\right]_\infty}+\frac{\left[-q^{-4j},-q^{-4j},q^{-12j},q^{-8j};q^{12j}\right]_\infty}{\left[-q^{-13j},q^{-13j},-q^{-2j};q^{12j}\right]_\infty}=0.$$

Dividing by $\left[-q^{j},q^{j},q^{7j},-q^{7j};q^{12j}\right]_\infty$ and using (2.2) to (2.7), we get

$$\left[\frac{\left[q^{3j},q^{9j};q^{12j}\right]_\infty}{[q^{j},q^{7j};q^{12j}]_\infty}-\frac{\left[-q^{3j},-q^{9j};q^{12j}\right]_\infty}{\left[-q^{j},-q^{7j};q^{12j}\right]_\infty}\right]$$
$$=\frac{2q^{-j}\left[-q^{4j},-q^{4j},q^{12j},q^{8j};q^{12j}\right]_\infty\left[q^{2j};q^{24j}\right]_\infty}{\left[-q^{-j},q^{-j},-q^{2j};q^{12j}\right]_\infty\left[q^{12j};q^{24j}\right]_\infty}$$

$$+\frac{2q^{j}\left[q^{2j},q^{2j},-q^{10j},-q^{6j};q^{12j}\right]_\infty\left[q^{2j};q^{24j}\right]_\infty}{\left[q^{j},-q^{j},-q^{2j};q^{12j}\right]_\infty\left[q^{12j};q^{24j}\right]_\infty}.$$

(3.6) $$R\left(q^{j}\right)-R\left(-q^{j}\right)=\frac{-2q^{j}\left[-q^{4j},-q^{4j},q^{12j},q^{8j};q^{12j}\right]_\infty\left[q^{2j};q^{24j}\right]_\infty}{\left[-q^{j},q^{j},-q^{2j};q^{12j}\right]_\infty\left[q^{12j};q^{24j}\right]_\infty}+\frac{2q^{j}\left[q^{2j},q^{2j},-q^{10j},-q^{6j};q^{12j}\right]_\infty\left[q^{2j};q^{24j}\right]_\infty}{\left[q^{j},-q^{j},-q^{2j};q^{12j}\right]_\infty\left[q^{12j};q^{24j}\right]_\infty},$$

$$\beta_1\left(q^{2j}\right)=\frac{1}{2}\left[R\left(q^{j}\right)-R\left(-q^{j}\right)\right],$$

(3.7) $$\beta_1(q^{2j})=-\frac{q^{j}\left[-q^{4j},-q^{4j},q^{12j},q^{8j};q^{12j}\right]_\infty\left[q^{2j};q^{24j}\right]_\infty}{\left[-q^{j},q^{j},-q^{2j};q^{12j}\right]_\infty\left[q^{12j};q^{24j}\right]_\infty}+\frac{q^{j}\left[q^{2j},q^{2j},-q^{10j},-q^{6j};q^{12j}\right]_\infty\left[q^{2j};q^{24j}\right]_\infty}{\left[q^{j},-q^{j},-q^{2j};q^{12j}\right]_\infty\left[q^{12j};q^{24j}\right]_\infty},$$

By using (3.5) and (3.7), we get

(3.8) $$R\left(q^{j}\right)=\alpha_1\left(q^{2j}\right)+\beta_1\left(q^{2j}\right)$$

$$=\frac{\left[q^{2j},q^{4j};q^{12j}\right]_\infty\left[q^{6j};q^{24j}\right]_\infty}{\left[q^{6j};q^{12j}\right]_\infty\left[q^{2j},q^{4j},q^{12j};q^{24j}\right]_\infty}+\frac{q^{2j}\left[q^{2j},q^{4j};q^{12j}\right]_\infty\left[q^{6j};q^{24j}\right]_\infty}{\left[q^{6j};q^{12j}\right]_\infty\left[q^{4j},q^{12j},q^{14j};q^{24j}\right]_\infty}$$
$$-\frac{q^{j}\left[-q^{4j},-q^{4j},q^{12j},q^{8j};q^{12j}\right]_\infty\left[q^{2j};q^{24j}\right]_\infty}{\left[-q^{j},q^{j},-q^{2j};q^{12j}\right]_\infty\left[q^{12j};q^{24j}\right]_\infty}$$
$$+\frac{q^{j}\left[q^{2j},q^{2j},-q^{10j},-q^{6j};q^{12j}\right]_\infty\left[q^{2j};q^{24j}\right]_\infty}{\left[q^{j},-q^{j},-q^{2j};q^{12j}\right]_\infty\left[q^{12j};q^{24j}\right]_\infty},$$

Now by applying (2.2) and (3.3),we obtain

$$\frac{[q^{3j},q^{9j};q^{12j}]}{[q^{j},q^{7j};q^{12j}]}=\frac{\left[q^{2j},q^{4j};q^{12j}\right]_\infty\left[q^{6j};q^{24j}\right]_\infty}{\left[q^{6j};q^{12j}\right]_\infty\left[q^{2j},q^{4j},q^{12j};q^{24j}\right]_\infty}+\frac{q^{2j}\left[q^{2j},q^{4j};q^{12j}\right]_\infty\left[q^{6j};q^{24j}\right]_\infty}{\left[q^{6j};q^{12j}\right]_\infty\left[q^{4j},q^{12j},q^{14j};q^{24j}\right]_\infty}$$
$$+\frac{q^{j}\left[q^{2j},q^{2j},-q^{10j},-q^{6j};q^{12j}\right]_\infty\left[q^{2j};q^{24j}\right]_\infty}{\left[q^{j},-q^{j},-q^{2j};q^{12j}\right]_\infty\left[q^{12j};q^{24j}\right]_\infty}$$

Now, To prove (3.2), we consider (3.4)

$$R\left(q^{j}\right)+R\left(-q^{j}\right)=\frac{2\left[q^{2j},q^{4j};q^{12j}\right]_\infty\left[q^{6j};q^{24j}\right]_\infty}{\left[q^{6j};q^{12j}\right]_\infty\left[q^{4j},q^{12j};q^{24j}\right]_\infty}\left[\frac{1}{\left[q^{2j};q^{24j}\right]_\infty}+\frac{q^{2j}}{\left[q^{14j};q^{24j}\right]_\infty}\right].$$

Multiplying (3.4) by $\dfrac{\left[q^{2j},q^{14j};q^{24j}\right]_\infty}{\left[q^{6j};q^{18j};q^{24j}\right]_\infty}$.

We get

$$(3.9)\quad R\left(q^{j}\right)^{-1}+R\left(-q^{j}\right)^{-1}=\frac{2\left[q^{2j},q^{4j};q^{12j}\right]_\infty\left[q^{6j};q^{24j}\right]_\infty\left[q^{2j},q^{14j};q^{24j}\right]_\infty}{\left[q^{6j};q^{12j}\right]_\infty\left[q^{4j},q^{12j};q^{24j}\right]_\infty\left[q^{6j},q^{18j};q^{24j}\right]_\infty}\times\left[\frac{1}{\left[q^{2j};q^{24j}\right]_\infty}+\frac{q^{2j}}{\left[q^{14j};q^{24j}\right]_\infty}\right]$$

$$\alpha_2\left(q^{2j}\right)=\frac{1}{2}\left[R\left(q^{j}\right)^{-1}+R\left(-q^{j}\right)^{-1}\right],$$

$$(3.10)\quad \alpha_2(q^{2j})=\frac{\left[q^{2j},q^{4j};q^{12j}\right]_\infty\left[q^{2j},q^{14j};q^{24j}\right]_\infty}{\left[q^{6j};q^{12j}\right]_\infty\left[q^{2j},q^{4j},q^{6j},q^{12j};q^{24j}\right]_\infty}+\frac{q^{2j}\left[q^{2j},q^{4j};q^{12j}\right]_\infty\left[q^{2j},q^{14j};q^{24j}\right]_\infty}{\left[q^{6j};q^{12j}\right]_\infty\left[q^{4j},q^{6j},q^{12j},q^{14j};q^{24j}\right]_\infty}.$$

Now, let

$$\beta_2(q^{2j})=\frac{1}{2}[R\left(q^{j}\right)^{-1}-R\left(-q^{j}\right)^{-1}]$$

Multiplying (3.6) by

$$\frac{\left[q^{2j},q^{14j};q^{24j}\right]_\infty}{\left[q^{6j};q^{18j};q^{24j}\right]_\infty}$$ , we get

$$(3.11)\quad \beta_2(q^{2j})=\frac{q^{j}\left[-q^{4j},-q^{4j},q^{12j},q^{8j};q^{12j}\right]_\infty\left[q^{2j},q^{2j},q^{14j};q^{24j}\right]_\infty}{\left[-q^{j},q^{j},-q^{2j};q^{12j}\right]_\infty\left[q^{6j},q^{12j},q^{18j};q^{24j}\right]_\infty}-\frac{q^{j}\left[q^{2j},q^{2j},-q^{10j},-q^{6j};q^{12j}\right]_\infty\left[q^{2j},q^{2j},q^{14j};q^{24j}\right]_\infty}{\left[-q^{j},q^{j},-q^{2j};q^{12j}\right]_\infty\left[q^{6j},q^{12j},q^{18j};q^{24j}\right]_\infty}$$

By using (3.10) and (3.11) we get (3.2)

$$R\left(q^{j}\right)^{-1}=\alpha_2(q^{2j})+\beta_2(q^{2j}),$$

$$(3.12)\qquad R(q^j)^{-1}=\frac{\left[q^{2j},q^{4j};q^{12j}\right]_\infty\left[q^{2j},q^{14j};q^{24j}\right]_\infty}{\left[q^{6j};q^{12j}\right]_\infty\left[q^{2j},q^{4j},q^{6j},q^{12j};q^{24j}\right]_\infty}+\frac{q^{2j}\left[q^{2j},q^{4j};q^{12j}\right]_\infty\left[q^{2j},q^{14j};q^{24j}\right]_\infty}{\left[q^{6j};q^{12j}\right]_\infty\left[q^{4j},q^{6j},q^{12j},q^{14j};q^{24j}\right]_\infty}+\frac{q^{j}\left[-q^{4j},-q^{4j},q^{12j},q^{8j};q^{12j}\right]_\infty\left[q^{2j},q^{2j},q^{14j};q^{24j}\right]_\infty}{\left[-q^{j},q^{j},-q^{2j};q^{12j}\right]_\infty\left[q^{6j},q^{12j},q^{18j};q^{24j}\right]_\infty}-\frac{q^{j}\left[q^{2j},q^{2j},-q^{10j},-q^{6j};q^{12j}\right]_\infty\left[q^{2j},q^{2j},q^{14j};q^{24j}\right]_\infty}{\left[-q^{j},q^{j},-q^{2j};q^{12j}\right]_\infty\left[q^{6j},q^{12j},q^{18j};q^{24j}\right]_\infty},$$

By applying (2.2) and (3.3),we obtain

$$\frac{[q^{j},q^{7j};q^{12j}]_\infty}{[q^{3j},q^{9j};q^{12j}]_\infty}=\frac{\left[q^{2j},q^{4j};q^{12j}\right]_\infty\left[q^{2j},q^{14j};q^{24j}\right]_\infty}{\left[q^{6j};q^{12j}\right]_\infty\left[q^{2j},q^{4j},q^{6j},q^{12j};q^{24j}\right]_\infty}+\frac{q^{2j}\left[q^{2j},q^{4j};q^{12j}\right]_\infty\left[q^{2j},q^{14j};q^{24j}\right]_\infty}{\left[q^{6j};q^{12j}\right]_\infty\left[q^{4j},q^{6j},q^{12j},q^{14j};q^{24j}\right]_\infty}-\frac{q^{j}\left[q^{2j},q^{2j},-q^{10j},-q^{6j};q^{12j}\right]_\infty\left[q^{2j},q^{2j},q^{14j};q^{24j}\right]_\infty}{\left[-q^{j},q^{j},-q^{2j};q^{12j}\right]_\infty\left[q^{6j},q^{12j},q^{18j};q^{24j}\right]_\infty}.$$

## 4. Special cases

(i) Substituting j=1in main result (3.1),we get

$$(4.1)\qquad \frac{\left[q^{3},q^{9};q^{12}\right]_\infty}{\left[q,q^{7};q^{12}\right]_\infty}=\frac{\left[q^{2},q^{4};q^{12}\right]_\infty\left[q^{6};q^{24}\right]_\infty}{\left[q^{6};q^{12}\right]_\infty\left[q^{2},q^{4},q^{12};q^{24}\right]_\infty}+\frac{q^{2}\left[q^{2},q^{4};q^{12}\right]_\infty\left[q^{6};q^{24}\right]_\infty}{\left[q^{6};q^{12}\right]_\infty\left[q^{4},q^{12},q^{14};q^{24}\right]_\infty}+\frac{q\left[q^{2},q^{2},-q^{10},-q^{6};q^{12}\right]_\infty\left[q^{2};q^{24}\right]_\infty}{\left[-q,q,-q^{2};q^{12}\right]_\infty\left[q^{12};q^{24}\right]_\infty}.$$

By applying (2.5) in (4.1), we obtain

(4.2)
$$\frac{\left[q^3;q^6\right]_\infty}{\left[q;q^6\right]_\infty}=\frac{\left[q^2,q^4;q^{12}\right]_\infty\left[q^6;q^{24}\right]_\infty}{\left[q^6;q^{12}\right]_\infty\left[q^2,q^4,q^{12};q^{24}\right]_\infty}+\frac{q^2\left[q^2,q^4;q^{12}\right]_\infty\left[q^6;q^{24}\right]_\infty}{\left[q^6;q^{12}\right]_\infty\left[q^4,q^{12},q^{14};q^{24}\right]_\infty}+\frac{q\left[q^2,q^2,-q^{10},-q^6;q^{12}\right]_\infty\left[q^2;q^{24}\right]_\infty}{\left[-q,q,-q^2;q^{12}\right]_\infty\left[q^{12};q^{24}\right]_\infty},$$

(ii) Substituting j=1 in main result (3.2), we get

(4.3)
$$\frac{[q,q^7;q^{12}]_\infty}{[q^3,q^9q^{12}]_\infty}=\frac{\left[q^2,q^4;q^{12}\right]_\infty\left[q^2,q^{14};q^{24}\right]_\infty}{\left[q^6;q^{12}\right]_\infty\left[q^2,q^4,q^6,q^{12};q^{24}\right]_\infty}+\frac{q^2\left[q^2,q^4;q^{12}\right]_\infty\left[q^2,q^{14};q^{24}\right]_\infty}{\left[q^6;q^{12}\right]_\infty\left[q^4,q^6,q^{12},q^{14};q^{24}\right]_\infty}-\frac{q\left[q^2,q^2,-q^{10},-q^6;q^{12}\right]_\infty\left[q^2,q^2,q^{14};q^{24}\right]_\infty}{\left[-q,q,-q^2;q^{12}\right]_\infty\left[q^6,q^{12},q^{18};q^{24}\right]_\infty}.$$

By applying (2.5) in (4.3), we obtain

(4.4)
$$\frac{[q;q^6]_\infty}{[q^3;q^6]_\infty}=\frac{\left[q^2,q^4;q^{12}\right]_\infty\left[q^2,q^{14};q^{24}\right]_\infty}{\left[q^6;q^{12}\right]_\infty\left[q^2,q^4,q^6,q^{12};q^{24}\right]_\infty}+\frac{q^2\left[q^2,q^4;q^{12}\right]_\infty\left[q^2,q^{14};q^{24}\right]_\infty}{\left[q^6;q^{12}\right]_\infty\left[q^4,q^6,q^{12},q^{14};q^{24}\right]_\infty}-\frac{q\left[q^2,q^2,-q^{10},-q^6;q^{12}\right]_\infty\left[q^2,q^2,q^{14};q^{24}\right]_\infty}{\left[-q,q,-q^2;q^{12}\right]_\infty\left[q^6,q^{12},q^{18};q^{24}\right]_\infty},$$

(iii) Substituting j=2 in (3.1) and applying (2.2), we get

$$(4.5)\qquad \frac{[q^6,q^{18};q^{24}]_\infty}{[q^2,q^{14};q^{24}]_\infty}=\frac{\left[q^4,q^8;q^{24}\right]_\infty\left[q^{12};q^{48}\right]_\infty}{\left[q^{12};q^{24}\right]_\infty\left[q^4,q^8,q^{24};q^{48}\right]_\infty}$$
$$+\frac{q^4\left[q^4,q^8;q^{24}\right]_\infty\left[q^{12};q^{48}\right]_\infty}{\left[q^{12};q^{24}\right]_\infty\left[q^8,q^{24},q^{20};q^{48}\right]_\infty}$$
$$+\frac{q^2\left[q^4,q^4,-q^{20},-q^{12};q^{24}\right]_\infty\left[q^4;q^{48}\right]_\infty}{\left[-q^2,q^2,-q^4;q^{24}\right]_\infty\left[q^{24};q^{48}\right]_\infty}.$$

(iv) Substituting j=2 in (3.2) and applying (2.2) , we get

$$(4.6)\qquad \frac{[q^2,q^{14};q^{24}]_\infty}{[q^6,q^{18};q^{24}]_\infty}=\frac{\left[q^4,q^8;q^{24}\right]_\infty\left[q^4,q^{20};q^{48}\right]_\infty}{\left[q^{12};q^{24}\right]_\infty\left[q^4,q^8,q^{12},q^{24};q^{28}\right]_\infty}$$
$$+\frac{q^4\left[q^4,q^8;q^{24}\right]_\infty\left[q^4,q^{28};q^{48}\right]_\infty}{\left[q^{12};q^{24}\right]_\infty\left[q^8,q^{12},q^{24},q^{20};q^{48}\right]_\infty}$$
$$-\frac{q^2\left[q^4,q^4,-q^{20},-q^{12};q^{24}\right]_\infty\left[q^4,q^4,q^{20};q^{48}\right]_\infty}{\left[-q^2,q^2,-q^4;q^{24}\right]_\infty\left[q^{12},q^{12},q^{24};q^{48}\right]_\infty}.$$

(v) By taking j=2 in (3.3) and applying (2.2) and (2.6), we get relation between R(q) and $R(q^2)$.

$$(4.4)\qquad R(q^2)=\frac{\left[q^6,q^{18};q^{24}\right]_\infty}{\left[q^2,q^{14};q^{24}\right]_\infty}=\frac{[q^3,-q^3q^9,-q^9;q^{12}]_\infty}{[q,-q,q^7,-q^7;q^{12}]_\infty}=R(q)R(-q).$$

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