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Conformal- Kropina Change of Finsler Metric

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Abstract: The purpose of the present paper is to find the necessary and sufficient conditions under which a Conformal-Kropina change becomes a Projective change. The condition under which a Conformal-Kropina change of Douglas space becomes a Douglas space have been also found.
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1. Introduction

Let $F^n = \left(M^n, L\right)$ is a Finsler space, where L is fundamental function of x and $y = \dot{x}$ defined on smooth manifold M^n. In the paper[1] the necessary and sufficient condition under which a Kropina change becomes a Projective change. Let $\beta = b_i\left(x\right)y^i$ be one form on the manifold M^n, $L \to e^{\sigma}\dfrac{L^z}{\beta}$ is called conformal – Kropina change of Finsler metric L. If we write

(1.1) $$\bar{L} \to e^{\sigma}\frac{L^2}{\beta},$$

where σ is scalar function of x. Then Finsler space $\bar{F}^n = \left(M^n, \bar{L}\right)$ is said to be obtained from Finsler space $F^n = \left(M^n, L\right)$ by Conformal-Kropina change. The quantities corresponding to $\bar{F}^n$ is denoted by putting bar on those quantities. Some basic tensor of $F^n = \left(M^n, L\right)$ are given as follows:

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y_i \partial y_j} \qquad l_i = \frac{\partial L}{\partial y^i} = L_i \ and \quad h_{ij} = g_{ij} - l_i\, l_j$$

where g_{ij} is fundamental metric tensor, l_i is normalized element of support and h_{ij} is angular metric tensor.

Partial derivative with respect to x^i and y^i will be denoted as ∂_i and $\dot{\partial}$ respectively and derivatives are written as

(1.2) $$L_i = \frac{\partial L}{\partial y^i},\ L_{ij} = \frac{\partial^2 L}{\partial y^j \partial y^i} \ and \quad L_{ijk} = \frac{\partial^3 L}{\partial y^k \partial y^j \partial y^i}.$$

The equation of geodesic of a Finsler space[2] is

$$\frac{d^2 x^i}{ds^2} + \frac{d^2 x^i}{ds^2} + 2G^i\left(x, \frac{dx}{ds}\right) = 0,$$

where G^i is positively homogeneous of degree two in y^i and is given by

$$2G^i = \frac{g^{ij}}{2}\left(y^r \bar{\partial}\partial_r L^2 - \partial_j L^2\right).$$

Berwald connection $B\Gamma = \left(G^i{}_{jk}, G^i{}_j, 0\right)$ of Finler space $F^n = \left(M^n, L\right)$ is given by[2]

$$G_j^{\ i} = \frac{\partial G^i}{\partial y^j}, \quad G^i{}_{jk} = \frac{\partial G_j^{\ i}}{\partial y^k}$$

Cartan connection $C\Gamma = (F_{jk}^{\ \ i}, G_j^{\ i}, C_{jk}^{\ \ i})$ is constructed from L with the help of following axioms[3]:

(i) Cartan connection $C\Gamma$ is v-metrical,
(ii) Cartan connection $C\Gamma$ is h-metrical,
(iii) The (v)v torsion tensor field S^1 of Cartan connection vanishes,
(iv) The (h)h torsion tensor field T of Cartan connection vanishes,
(v) The deflection Tensor field D of Cartan connection vanishes.

h and v- covariant derivative with respect to Cartan connection are denoted by |k and |k.
Let

(1.3) $$\overline{G^{i}} = G^{i} + D^{i},$$

where D^i is difference tensor homogeneous of second degree in y^i.
Then $\overline{G_j^{\ i}} = G_j^{\ i} + D_j^{\ i}$,, $\overline{G_{jk}^{\ \ i}} = G_{jk}^{\ \ i} + D_{jk}^{\ \ i}$, , where $D^i_{\ j} = \dfrac{\partial D^i}{\partial y^j}$ and

$$D^i_{\ j_k} = \frac{\partial D_j^{\ i}}{\partial y^j}$$

are homogeneous of degree 1 and 0 in y^i respectively.

2. Difference Tensor D^j

From (1.1) and (1.2) we have

(2.1) $$\overline{L_i} = e^{\sigma}\left(\frac{2L}{\beta}L_i - \frac{L^2}{\beta^2}b_i\right), \ ,$$

(2.2) $$\overline{L_{ij}} = e^{\sigma}\left\{\frac{2L}{\beta}L_{ij} + \frac{2}{\beta}L_iL_j - \frac{2L}{\beta^2}\left(L_ib_j + L_jb_i\right) + \frac{2L^2}{\beta^3}b_ib_j\right\},$$

(2.3) $$\overline{L_{ijk}} = e^{\sigma}\left\{\begin{array}{l}\dfrac{2L}{\beta}L_{ijk} + \dfrac{2}{\beta}\left(L_{ij}L_k + L_{ik}L_j + L_{jk}L_i\right) - \dfrac{2L}{\beta^2}\left(L_{ij}b_k + L_{ik}b_j + L_{jk}b_i\right) \\ -\dfrac{2}{\beta^2}(L_{ij}L_k + L_{ik}L_j + L_{jk}L_i)\end{array}\right\}$$

(2.4) $$\partial_j\overline{L_i} = e^{\sigma}\left\{\begin{array}{l}\dfrac{2L}{\beta}\partial_jL_i + \left(\dfrac{2L}{\beta}L_i - \dfrac{L^2}{\beta^2}b_i\right)\partial_j\sigma + \left(\dfrac{2}{\beta}L_i - \dfrac{2L}{\beta^2}b_i\right)\partial_jL \\ +\left(\dfrac{2L^2}{\beta^3}b_i - \dfrac{2L}{\beta^2}L_i\right)\partial_j\beta - \dfrac{L^2}{\beta^2}\partial_jb_i\end{array}\right\}$$

(2.5)
$$\partial_k \overline{L_{ij}} = e^{\sigma}\left\{\begin{array}{l} \left(\frac{2L}{\beta}L_{ij}+\frac{2}{\beta}L_iL_j-\frac{2L}{\beta^2}\left(L_ib_j+L_jb_i\right)+\frac{2L^2}{\beta^3}b_ib_j\right)\partial_k\sigma+\frac{2L}{\beta}\partial_kL_{ij} \\ +\left(\frac{2}{\beta}L_{ij}-\frac{2}{\beta^2}\left(L_ib_j+L_jb_i\right)+\frac{4L}{\beta^3}b_ib_j\right)\partial_kL \\ +\left(\frac{4L}{\beta^3}\left(L_ib_j+L_jb_i\right)-\frac{6L^2}{\beta^4}b_ib_j-\frac{2L}{\beta^2}L_{ij}-\frac{2}{\beta^2}L_iL_j\right)\partial_k\beta \\ +\left(\frac{2}{\beta}L_j-\frac{2L}{\beta^2}b_j\right)\partial_kL_i+\left(\frac{2}{\beta}L_i-\frac{2L}{\beta^2}b_i\right)\partial_kL_j \\ +\left(\frac{2L^2}{\beta^3}b_i-\frac{2L}{\beta^2}L_i\right)\partial_kb_j+\left(\frac{2L^2}{\beta^3}b_j-\frac{2L}{\beta^2}L_j\right)\partial_kb_i \end{array}\right\}$$

Now in $\bar{F}^n$ and F^n, we have

(2.6)
$$\overline{L_{ijlk}} = 0 \Rightarrow \partial_k \overline{L_{ij}} - \overline{L_{ijr}}\,\overline{G_k}^r - \overline{L}_{ir}\,\overline{F_{ik}}^r = 0.$$

(2.7)
$$L_{ijlk} = 0 \Rightarrow \partial_k L_{ij} - L_{ijr}G_k^r - L_{ir}F_{jk}^r - L_{jr}F_{ik}^r = 0.$$

$$\overline{G_k}^r = G_k^r + D_k^r \quad and \quad \overline{F}_{ik}^r = F_{ik}^r + D_{ik}^{*r}$$

Putting the value from (2.2), (2.3), (2.5) and (2.7) in (2.6) and contract the resulting equation by y^k, we have

(2.8)
$$\left(\frac{4L}{\beta^3}\left(L_ib_j+L_jb_i\right)-\frac{6L^2}{\beta^4}b_ib_j-\frac{2L}{\beta^2}L_{ij}-\frac{2}{\beta^2}L_iL_j\right)r_{00}$$
$$+\left(\frac{2L^2}{\beta^3}b_i-\frac{2L}{\beta^2}L_i\right)\left(r_{j0}+s_{j0}\right)+\left(\frac{2L^2}{\beta^3}b_j-\frac{2L}{\beta^2}L_j\right)\left(r_{i0}+s_{i0}\right)$$
$$+\left(\frac{2L}{\beta}L_{ij}+\frac{2}{\beta}L_iL_j-\frac{2L}{\beta^2}\left(L_ib_j+L_jb_i\right)+\frac{2L^2}{\beta^3}b_ib_j\right)\Delta_1$$
$$-e^{-\sigma}\left(2\overline{L}_{ijr}D^r+\overline{L}_{ir}D_j^r+\overline{L}_{jr}D_i^r\right)=0,$$

where $\Delta_1 = \partial_k \sigma y^k$

Now in $\bar{F}^n$ and F^n , we have

(2.9) $$\bar{L}_{i|j} = 0 \Rightarrow \partial_j \bar{L}_i - \bar{L}_{ir} \bar{G}_j^r - \bar{L}_r \bar{F}_{ij}^r = 0$$

(2.10) $$L_{i|j} = 0 \Rightarrow \partial_j L_i - L_{ir} G_j^r - L_r F_{ij}^r = 0$$

Putting the value from (2.1), (2.2), (2.4) and (2.10) in (2.9), we have

(2.11) $$\frac{L^2}{\beta^2} b_{i|j} = e^{-\sigma} \bar{L}_{ir} D_j^r + e^{-\sigma} \bar{L}_r D_{ij}^{*r} - \left(\frac{2L}{\beta} L_i - \frac{L^2}{\beta^2} b_i \right) \partial_j \sigma - \left(\frac{2L^2}{\beta^3} b_i - \frac{2L}{\beta^2} L_i \right) \left(r_{j0} + s_{j0} \right)$$

Since

(2.12) $$2r_{ij} = b_{i|j} + b_{j|i}$$

Therefore putting the value from (2.11) in (2.12), we have

(2.13) $$\frac{2L^2}{\beta^2} r_{ij} = e^{-\sigma} \bar{L}_{ir} D_i^r + 2e^{-\sigma} \bar{L}_r D_{ij}^{*r} - \left(\frac{2L}{\beta} L_i - \frac{L^2}{\beta^2} b_i \right) \partial_j \sigma - \left(\frac{2L^2}{\beta^3} b_i - \frac{2L}{\beta^2} L_i \right) \left(r_{j0} + s_{j0} \right) - \left(\frac{2L}{\beta} L_j - \frac{L^2}{\beta^2} b_j \right) \partial_i \sigma - \left(\frac{2L^2}{\beta^3} b_j - \frac{2L}{\beta^2} L_i \right) \left(r_{i0} + s_{i0} \right)$$

Substract (2.8) from (2.13) and contract the resulting equation by $y^i y^j$, we get

(2.14) $$4L\beta L_r D^r - 2L^2 b_r D^r = -L^2 r_{00} + L^2 \beta \Delta_1$$

Since,

(2.15) $$2s_{ij} = b_{i|j} - b_{j|i}$$

Therefore putting the value from (2.11) in (2.15), we have

(2.16)
$$-\frac{2L^2}{\beta^2}s_{ij} = e^{-\sigma}\overline{L}_{ir}D_j^r - e^{-\sigma}\overline{L}_{jr}D_i^r - \left(\frac{2L}{\beta}L_i - \frac{L^2}{\beta^2}b_i\right)\partial_j\sigma$$
$$-\left(\frac{2L^2}{\beta^3}b_i - \frac{2L}{\beta^2}L_i\right)(r_{j0}+s_{j0}) + \left(\frac{2L}{\beta}L_j - \frac{L^2}{\beta^2}b_j\right)\partial_i\sigma$$
$$+\left(\frac{2L^2}{\beta^3}b_j - \frac{2L}{\beta^2}L_j\right)(r_{i0}+s_{i0})$$

Substract (2.8) from (2.16) and contract the resulting equation by $y^j b^i$, we have

(2.17)
$$-4L\beta b^2 L_r D^r + 4L^2 b^2 b_r D^r = -2L^2\beta s_0$$
$$+2r_{00}\left(L^2b^2 - \beta^2\right) + \left(2\beta^3 - L^2b^2\beta\right)\Delta_1 - \beta^2 L^2 \Delta_2$$

where $\Delta_2 = \partial_k \sigma b^k$

Solution of algebraic equation (2.14) and (2.17) is given by

(2.18) $$b_r D^r = \frac{1}{2b^2L^2}\left\{\left(b^2L^2 - 2\beta^2\right)r_{00} - 2\beta L^2 s_0 + 2\beta^3\Delta_1 - \beta^2 L^2 \Delta_2\right\}$$

(2.19) $$L_r D^r = \frac{1}{2b^2L^2}\left\{-L^3 s_0 - L\beta r_{00} + \left(L\beta^2 + \frac{b^2L^3}{2}\right)\Delta_1 - \frac{\beta L^3 \Delta_2}{2}\right\}$$

Substract (2.8) from (2.16) and contract the resulting equation by y^j , we have

(2.20)
$$-\frac{2L^2}{\beta^2}s_{i0} + \left(\frac{2L}{\beta}L_i - \frac{L^2}{\beta^2}b_i\right)\Delta_1 + \left(\frac{2L^2}{\beta^3}b_i - \frac{2L}{\beta^2}L_i\right)r_{00} - \frac{L^2}{\beta^2}\partial_i\sigma = e^{-\sigma}\overline{L}_{ir}D^r$$

Putting the value from (2.2) in (2.20)

using $LL_{ir} = g_{ir} - L_i L_r, L_i = l_i$ and contracting the resulting equation by g^{ij} , we have

$$-\frac{2L^2}{\beta^2}s_0{}^j+\left(\frac{2L}{\beta}l^j-\frac{L^2}{\beta^2}b^j\right)\triangle_1+\left(\frac{2L^2}{\beta^3}b^j-\frac{2L}{\beta^2}l^j\right)r_{00}$$

$$-\frac{L^2}{\beta}\partial_i\sigma g^{ij}=\frac{4}{\beta}D^j-\frac{4L}{\beta^2}l^jb_rD^r-\frac{4L}{\beta^2}b^jL_rD^r+\frac{4L}{\beta^2}b^jb_rD^r$$

or

$$\frac{4}{\beta}D^j=-\frac{2L^2}{\beta^2}s_0{}^j+l^j\left(\frac{2L}{\beta}\triangle_1-\frac{2L}{\beta^2}r_{00}+\frac{4L}{\beta^2}b_rD^r\right)$$
$$+b^j\left(\frac{2L^2}{\beta^3}r_{00}-\frac{L^2}{\beta^2}\triangle_1+\frac{4L}{\beta^2}L_rD^r-\frac{4L^2}{\beta^3}b_rD^r\right)-\frac{L^2}{\beta}\partial_i\sigma g^{ij}$$

or

(2.21) $$D^j=-\frac{L^2}{2\beta}s_0{}^j+\frac{y^j}{2\beta^2}\left(\beta^2\triangle_1-\beta r_{00}+2\beta b_rD^r\right)$$
$$+\frac{b^j}{2\beta^2}\left(L^2r_{00}-\frac{\beta L^2\triangle_1}{2}+2L\beta L_rD^r-2L^2b_rD^r\right)-\frac{L^2}{4}\partial_i\sigma g^{ij}$$

Proposition 2.1: Difference tensor of conformal kropina change of Finsler metric L is given by equations (2.21), (2.19) and (2.18).

3. Projective Change of Finsler Metric

Definition3.1[4]: A Finsler space $\bar{F}_n$ is called projective to Finsler space F^n if there is geodesics correspond between $\bar{F}^n$ and F^n.That is, $L\rightarrow\bar{L}$ is projective if $\bar{G}^i=\bar{G}^i+P(x,y)y^i$, where $P(x,y)$ is called projective factor, this is homogeneous scalar function of degree one in y^i

Putting $D^j=Py^j$ in equation (2.21), where P is projective factor and contracting the resulting equation by y_j , we have

(3.1) $$P=\frac{L_rD^r}{L}.$$

Putting the value from (2.19) in (3.1), we have

(3.2) $$P=\frac{1}{2b^2L^2}\left\{-L^2s_0-\beta r_{00}+\left(\beta^2+\frac{b^2L^2}{2}\right)\triangle_1-\frac{\beta L^2\triangle_2}{2}\right\}.$$

Putting the value from (3.2), (2.19) and (2.18) in (2.21), we have

$$(3.3)\quad r_{00} = -\frac{\beta s_0}{\Delta} - \frac{\beta^2 \Delta_2}{2\Delta} - \left(\frac{\beta b^2}{2} - \frac{\beta^3}{L^2}\right)\frac{\Delta_1}{\Delta},$$

where $\Delta = \left(\frac{\beta}{L}\right)^2 - b^2$

Putting the value from (3.3) in (3.2), we have

$$(3.4)\quad P = \frac{s_0}{2\Delta} + \frac{\beta}{4\Delta}\Delta_2 - \frac{b^2}{4\Delta}\Delta_1 .$$

Putting the value from (3.4), (3.3), (2.19) and (2.18) in equation (2.21), we have

$$(3.5)\quad s_0^{\ j} = b^j\left(-\frac{s_0}{\Delta} - \frac{\beta\Delta_2}{2\Delta} + \frac{\beta^2\Delta_1}{2L^2\Delta}\right) + y^j\left(\frac{\beta s_0}{L^2\Delta} + \frac{\beta^2\Delta_2}{2L^2\Delta} - \frac{\beta b^2\Delta_1}{2L^2\Delta}\right) - \frac{\beta}{2}\partial_i\sigma g^{ij}.$$

Equation (3.3) and (3.5) are necessary condition for conformal kropina change of Finsler metric to be projective.

Conversely, if condition (3.3) and (3.5) are satisfied, then put these value in (2.21), we have

$$D^j = \left(\frac{s_0}{2\Delta} + \frac{\beta}{4\Delta}\Delta_2 - \frac{b^2}{4\Delta}\Delta_1\right)y^j = Py^j \cdot$$

That is $\bar{F}^n$ is projective to F^n.

Theorem 3.1: *The Conformal- Kropina change of Finsler space is projective iff equation (3.3) and (3.5) are satisfied and then projective factor P is given by* $P = \left(\frac{s_0}{2\Delta} + \frac{\beta}{4\Delta}\Delta_2 - \frac{b^2}{4\Delta}\Delta_1\right)$

If we put $\sigma = 0$, we find result that has been discussed in[1].

4. Douglas Space

Definition 4.1[5]: A Finsler space F^n is called Douglas space if $G^i y^j - G^j y^i$ is homogeneous polynomial of degree three in y^i. In brief, homogeneous polynomial of degree r in y^i is denoted by hp(r).
If we denote

(4.1) $$B^{ij} = D^i y^j - D^j y^i$$

from equation (2.21), we have

(4.2) $$B^{ij} = \left(L^2 r_{00} - \frac{\beta L^2 \Delta_1}{2} + 2L\beta L_r D^r - 2L^2 b_r D^r \right) \frac{\left(b^i y^j - b^j y^i\right)}{2\beta^2} - \frac{L^2}{2\beta}\left(s_0{}^i y^j - s_0{}^j y^i\right) - \frac{L^2}{4}\left(\partial_p \sigma g^{ip} y^j - \partial_p \sigma g^{jp} y^i\right)$$

From (4.2), we see that B^{ij} is hp (3).
That is, if Douglas space is transformed to be Douglas space by Conformal-Kropina change of Finsler metric, then B^{ij} is hp(3) and if B^{ij} is hp(3)then Douglas space transformed by Conformal-Kropina change is Douglas space.

Theorem 4.1: *The conformal – Kropina change of Douglas space is Douglas space iff* B^{ij} *given by (4.2) is hp(3).*

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