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Fixed Point Theorems under New Commuting Conditions

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Abstract. The aim of the present paper is to employ new commuting conditions to obtain some fixed point theorems as applications of the new notions. The new definitions are proper generalisations of the existing notions of similar type.

Key words: α-compatible maps, Compatible maps, noncompatible maps, R-commuting maps, quasi α-compatible maps, quasi R-commuting maps.
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1. Introduction

Recently, the authors[1] unified and generalized the notions of R-weak commuting and compatibility and their various analogues by introducing the notions of quasi R-commuting and quasi α-compatible. Two self-maps f and g of a metric space (X,d) are called compatible [2] if $\lim_n d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X. Two self-mappings f and g of a metric space (X,d) are called R-weakly commuting[3] if there exists some real number $R > 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$ for all x in X. The self-maps f and g are called R-weakly commuting of type A_f (of type A_g) if there exists some number $R > 0$ such that $d(fgx, ggx) \leq Rd(fx, gx)(d(ffx, gfx) \leq Rd(fx, gx))$ for all x in X [4]. The self- maps f and g are called g-compatible (f- compatible) if $\lim_n d(ffx_n, gfx_n) = 0(\lim_n d(fgx_n, ggx_n) = 0)$ whenever $\{x_n\}$ is a sequence in X

such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X [5]. The mappings f and g are defined to be compatible of type (P) if $\lim_n d(ffx_n, ggx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X [6]. If f and g are selfmaps of a metric space (X,d) and if $\{x_n\}$ is a sequence in X such that $fx_n = gx_{n+1}, n = 0,1,2,....,$ then the set $O(x,f,g) = \{fx_n, n = 0,1,2,......\}$ is called the (f,g) orbit at x and g (or f) is called orbitally continuous at x [7] if g (or f) is continuous on the closure of $O(x,f,g)$.

In a recent work[1] the authors introduced the following notions:

Definition 1.1. Two selfmaps f and g of a metric space (X,d) are called R-commuting provided there exists a positive real number R such that $d(ffx, gfx) \le Rd(fx, gx), d(fgx, gfx) \le Rd(fx, gx), d(fgx, ggx) \le Rd(fx, gx)$ and $d(ffx, ggx) \le Rd(fx, gx)$ for each x in X.

Definition 1.2. Two selfmaps f and g of a metric space (X,d) are called quasi R - commuting provided there exists a real number $R > 0$ such that given x in X we have $d(ffx, gfx) \le Rd(fx, gx)$ or $d(fgx, gfx) \le Rd(fx, gx)$ or $d(fgx, ggx) \le Rd(fx, gx)$ or $d(ffx, ggx) \le Rd(fx, gx)$.

Example 1.1. Let $X = [1, \infty)$ equipped with the Euclidean metric. Define $f, g : X \to X$ by

$$fx = 2x - 1, \qquad gx = 3x - 2 \text{ for each } x \text{ in } X.$$

Then $d(ffx, gfx) \le 5d(fx, gx), d(fgx, gfx) \le 5d(fx, gx)$, $d(fgx, ggx) \le 5d(fx, gx)$ and $d(ffx, ggx) \le 5d(fx, gx)$ for each x. We thus see that the mappings f and g are R- commuting with R=5. It may be observed in this example that $\lim_n fx_n = \lim_n gx_n = t$ for some sequence $\{x_n\}$ in X implies $t = 1$, $x_n \to 1$, $\lim_n d(ffx_n, gfx_n) = 0$, $\lim_n d(fgx_n, gfx_n) = 0$, $\lim_n d(fgx_n, ggx_n) = 0$ and $\lim_n d(ffx_n, ggx_n) = 0$.

This motivates the following generalization of Definition 1.1:

Definition 1.3. Two selfmappings f and g of a metric space (X,d) are called α-compatible iff $\lim_n d(ffx_n, gfx_n) = 0$, $\lim_n d(fgx_n, gfx_n) = 0$,

$\lim_n d(fgx_n, ggx_n) = 0$ and $\lim_n d(ffx_n, ggx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.

If f and g are quasi R-commuting selfmaps of a metric space (X,d) and $\{x_n\}$ is any sequence in X then the definition of quasi R-commuting implies that the sequence $\{x_n\}$ will split up in at most four sub-sequences such that any of these sub-sequences, say $\{x_{n_i}\}$, satisfies at least one of the four conditions $d(ffx_{n_i}, gfx_{n_i}) \le Rd(fx_{n_i}, gx_{n_i})$, $d(fgx_{n_i}, gfx_{n_i}) \le Rd(fx_{n_i}, gx_{n_i})$, $d(fgx_{n_i}, ggx_{n_i}) \le Rd(fx_{n_i}, gx_{n_i})$ and $d(ffx_{n_i}, ggx_{n_i}) \le Rd(fx_{n_i}, gx_{n_i})$.
This observation motivates the following generalization of compatibility.

Definition 1.4. Two selfmappings f and g of a metric space (X,d) are called quasi α-compatible provided every sequence $\{x_n\}$ in X satisfying $\lim_n fx_n = \lim_n gx_n = t$ for some t in X splits up in at most four sub-sequences such that any of these sub-sequences, say $\{x_{n_i}\}$, satisfies at least one of the four conditions $\lim_{n_i} d(ffx_{n_i}, gfx_{n_i}) = 0$, $\lim_{n_i} d(fgx_{n_i}, gfx_{n_i}) = 0$, $\lim_{n_i} d(fgx_{n_i}, ggx_{n_i}) = 0$ and $\lim_{n_i} d(ffx_{n_i}, ggx_{n_i}) = 0$.

Example 1.2. Let $X = [0,20]$ and d be the Euclidean metric. Define $f, g : X \to X$ by
$fx = 2 + (2-x)/2$ if $x \le 2$, $fx = 6$ if $2 < x \le 5$, $fx = 2$ if $x > 5$,
$gx = 2$ if $x \le 2$, $gx = 12$ if $2 < x \le 5$, $gx = 2 + (x-5)/2$ if $x > 5$.
Then $d(fgx, ggx) \le d(fx, gx)$ for each $x \le 2$, $d(ffx, gfx) \le d(fx, gx)$ for each $x > 2$. We thus see that the mappings f and g are quasi R-commuting with *R=1* but the mappings are neither R-weakly commuting nor R-weakly commuting of type (A_g) nor R-weakly commuting of type (Af). It may also be noted that the mappings f and g are quasi α-compatible but are neither compatible nor f-compatible nor g-compatible nor compatible of type *(P)*.

2. Main Results

Theorem 2.1. *Let f and g be orbitally continuous selfmappings of a complete metric space (X,d) such that $fX \subseteq gX$ and*

$$(i)\ d(fx, fy) \le k\left[d(fx, gx) + d(fy, gy)\right],\ 0 \le k < 1/2.$$

If f and g are quasi R-commuting then f and g have a unique common fixed point.

Proof: Let x_0 be any point in X. Define sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_n = fx_n = gx_{n+1}$, $n = 0,1,2...$ .This can be done since $fX \subseteq gX$. Now using a standard argument and by virtue of (i) it follows that $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a point t in X such that $y_n \to t$ as $n \to \infty$. Also, $\lim_n fx_n = t$, $\lim_n gx_n = t$. These limits and orbital continuity of f and g imply that

(1) $$\lim_n fgx_n = \lim_n ffx_n = ft, \quad \lim_n ggx_n = \lim_n gfx_n = gt\ .$$

Now quasi R-commutativity of f and g implies that the sequence $\{x_n\}$ will split up in at most four sub-sequences such that any of these sub-sequences, say $\{x_{n_i}\}$, satisfies at least one of the four conditions

$$d\left(ffx_{n_i}, gfx_{n_i}\right) \le Rd\left(fx_{n_i}, gx_{n_i}\right), d\left(fgx_{n_i}, gfx_{n_i}\right) \le Rd\left(fx_{n_i}, gx_{n_i}\right),$$

$$d\left(fgx_{n_i}, ggx_{n_i}\right) \le Rd\left(fx_{n_i}, gx_{n_i}\right) \text{ and } d\left(ffx_{n_i}, ggx_{n_i}\right) \le Rd\left(fx_{n_i}, gx_{n_i}\right).$$

This, in view of *(1)* implies that $ft = gt$. Thus, quasi R-commutativity in combination with orbital continuity implies that t is a coincidence point of f and g. Now using (i) we get $d(fx_n, ft) \le k\left[d(fx_n, gx_n) + d(ft, gt)\right]$. On letting $n \to \infty$ this yields $t = ft = gt$. Hence t is a common fixed point of f and g. Uniqueness of the common fixed point follows from (i).

Since R-commuting and α-compatible are much stronger conditions than quasi R- commuting, there is a large class of mappings which are quasi R-commuting but not α-compatible. A remarkable feature of this class of mappings is that we can not only prove fixed point theorems under contractive conditions but also under Lipschitz type conditions. The next theorem demonstrates that for such mappings we can generalise Theorem

2.1 by replacing the Kannan type contractive condition (i) by a Lipschitz type condition in which the range of k is extended from $[0,1/2)$ to $[0,1)$.

Theorem 2.2. *Let f and g be selfmappings of a metric space (X,d) satisfying the conditions*

(i) $(fX)^c \subseteq gX$, $(fX)^c$ *being the closure of* fX ,

(ii) $d(fx, fy) \le k[d(fx, gx) + d(fy, gy)]$, $0 \le k < 1$.

(iii) $d(fx, f^2x) < d(gx, g^2x)$ *whenever* $gx \ne g^2x$.

If f and g are quasi R-commuting but not α-compatible then f and g have a common fixed point.

Proof. The mappings f and g are not α-compatible implies that there exists a sequence $\{x_n\}$ in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X and at least one of $\lim_n d(ffx_n, gfx_n)$, $\lim_n d(fgx_n, gfx_n)$, $\lim_n d(fgx_n, ggx_n)$ and $\lim_n d(ffx_n, ggx_n)$ is either non zero or nonexistent. Since $t \in (fX)^c$ and $(fX)^c \subseteq gX$ there exists a point u in X such that $t = gu$. By (ii) we now get

$$d(fx_n, fu) \le k[d(fx_n, gx_n) + d(fu, gu)].$$

On letting $n \to \infty$ we get $t = fu$, that is, $t = fu = gu$. Since f and g are quasi R-commuting, there exists $R > 0$ such that
$d(ffu, ggu) \le Rd(fu, gu)$
or $d(ffu, gfu) \le Rd(fu, gu)$ or $d(fgu, gfu) \le Rd(fu, gu)$ or
$d(fgu, ggu) \le Rd(fu, gu)$, that is, $ffu = fgu = gfu = ggu$. If $fu \ne ffu$ then by (iii) we get $d(fu, ffu) < d(gu, ggu) = d(fu, ffu)$, a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g .

Example 2.1. Let $X = [2,20]$ and d be the Euclidean metric. Define $f, g : X \to X$ by

$fx = 2$ if $x = 2$ or > 5, $fx = 6$ if $2 < x \le 5$,

$g2 = 2$, $gx = 11$ if $2 < x \le 5$, $gx = (x+1)/3$ if $x > 5$.

Then f and g satisfy the conditions of Theorem 2.2 and have a common fixed point $x=2$. It may be noted in this example that f and g satisfy the condition $d(fx,fy)\le(4/5)\left[d(fx,gx)+d(fy,gy)\right]$ for all x, y in X. The pair (f,g) is quasi R-commuting since $d(ffx,gfx)\le d(fx,gx)$ for each x in X. If we consider the sequence $\{x_n\}$ given by $x_n=5+(1/n)$, $n=1,2,3,...,$, then $\lim_n fx_n=\lim_n gx_n=2$ but $\lim_n d(fgx_n,gfx_n)=4\ne 0$, that is, f and g are not α-compatible.

Remark 2.1. Using the ideas contained in Theorem 2.2 we can not only generalize several well known fixed point theorems concerning non-compatible mappings (e.g. Imdad et al [8], Pant and Pant [9], Singh and Kumar [10]) but can also obtain their analogues for mappings which are quasi R-commuting but not f-compatible (or g-compatible or compatible type *(P)*).

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