

# Computational Aspects of Virus Propagation Models

**Vijay Kumar Nath Tiwari**

St. Xavier's School, Ranchi, Jharkhand

Email: vijaymaths1000@gmail.com

(Received December 24, 2015)

**Abstract:** Computer virus can damage the computer systems by easing data, stealing information, modifying the normal operation and so forth. In this scenario, the large number of existing computer virus and their high level of destructivity appear as an important risk factor for corporations and individuals. Developing a mathematical model for the computer virus propagation is of critical importance not only for better understanding of the behaviour of computer virus but also for stopping the spread of virus. This type of model leads to a better understanding and prediction of the scale and speed of computer virus propagation. Due to the high similarity between computer virus and biological virus, some models for the spread of computer virus have been proposed in literature. In this paper, we have discussed the condition for the formulation of virus propagation and also an attempt has been made to analyse the situation in which a stage may be reached when all the infected nodes are immediately removed.

**Keywords:** Malicious agents, Stochasticbehaviour, Susceptible nodes, Infectious nodes, Anti – malicious software, Recovery rate of infected nodes, Effective removal rate.

**2010 AMS Classification No.:** 93A30.

## 1. Introduction

In this model we have discussed the condition for the formulation of virus propagation in the system network. The virus propagation in the system does not build up if the rate of infectivity is negative. It builds up only when the effective removal rate is less than the initial number of susceptible nodes in the system and in this case all the nodes do not get infected in which a stage may be reached when all the infected nodes are immediately removed. Thus a virus propagation is build up only when the density of susceptible nodes is

high and the rate of removal is low due to inadequate isolation facilities during virus propagation. An infected node can be removed from the scene of the virus propagation or can be made immune by the use of anti malicious software in the system network.

## 2. Model Assumptions

1. The system under consideration is such that, initially at time  $t = 0$ , there are  $n$  susceptible nodes.
2. The infectious rate of susceptible nodes and recovery rate of infected nodes are assumed to be constant.
3. The number of infected nodes grows at the rate proportional to the difference of product of susceptible and infected nodes with  $\gamma I$ .
4. The number of susceptible nodes decreases at the rate proportional to the product of susceptible and infected nodes.
5. The number of removal nodes grows at the rate proportional to the number of infected nodes.

## Notations

$R$ : Removal nodes at any time  $t$ .

$S$ : Susceptible nodes at any time  $t$ .

$I$ : Infected nodes at any time  $t$ .

$N$ : Total number of nodes at time  $t = 0$ .

$\beta$ : Infectious rate of susceptible nodes.

$\gamma$ : Recovery rate of infected nodes.

$\rho$ : Effective removal rate.

## 3. Analysis of the Model

Let the infected nodes be removed from the system at a rate  $\gamma$  so that our system of equations becomes

$$(1) \quad \frac{ds}{dt} = -\beta SI,$$

$$(2) \quad \frac{dI}{dt} = \beta SI - \gamma I = \beta I \left( S - \frac{\gamma}{\beta} \right) = \beta I (S - \rho); \rho = \frac{\gamma}{\beta},$$

$$(3) \quad \frac{dR}{dt} = \gamma I,$$

with the initial conditions

$$(4) \quad S(0) = S_0 > 0, I(0) = I_0 > 0, R(0) = R_0 = 0, S_0 + I_0 = N = n + 1.$$

Equations (1), (2), (3) and (4) together imply

$$(5) \quad S(t) + I(t) + R(t) = N = S_0 + I_0.$$

From equations (1) and (3), we get

$$(6) \quad \frac{dS}{dR} = -\frac{\beta}{\gamma} S = -\frac{S}{\rho},$$

for  $\rho = \frac{\gamma}{\beta}$ ,  $\rho$  is called the effective removal rate.

Integrating the differential equation (6), we have

$$(7) \quad \log S = -\frac{R}{\rho} + K.$$

When  $S = S_0$ ,  $R = 0$ . Therefore  $\log S_0 = K$ .

Hence,

$$\log S = -\frac{R}{\rho} + \log S_0,$$

i.e.

$$(8) \quad S = S_0 e^{-\frac{R}{\rho}}.$$

In view of equations (5) and (8), equation (3) may be written as

$$\frac{dR}{dt} = \gamma \left( N - S_0 e^{\frac{R}{\rho}} - R \right).$$

Integrating the above equation, we have

$$(9) \quad t = \frac{1}{\gamma} \int \frac{dR}{N - S_0 e^{\frac{R}{\rho}} - R}.$$

### Approximate Solution of the Model

Putting  $e^{\frac{R}{\rho}} = 1 - \frac{R}{\rho} + \frac{R^2}{2\rho^2}$  in (9), we have

$$\begin{aligned} t &= \frac{1}{\gamma} \int \frac{dR}{N - S_0 \left( 1 - \frac{R}{\rho} + \frac{R^2}{2\rho^2} \right) - R} \\ &= \frac{-2\rho^2}{\gamma S_0} \int \frac{dR}{\frac{-2\rho^2 N}{S_0} + 2\rho^2 - 2\rho R + \frac{2\rho^2 R}{S_0} + R^2} \\ &= \frac{-2\rho^2}{\gamma S_0} \int \frac{dR}{R^2 - 2\rho \left( 1 - \frac{\rho}{S_0} \right) R + 2\rho^2 - \frac{2N\rho^2}{S_0}}. \end{aligned}$$

Putting  $\rho \left( 1 - \frac{\rho}{S_0} \right) = K_1$  and  $2\rho^2 - \frac{2N\rho^2}{S_0} = K_2$ , we get

$$\begin{aligned} t &= -\frac{2\rho^2}{\gamma S_0} \int \frac{dR}{R^2 - 2K_1 R + K_2} \\ &= \frac{2\rho^2}{\gamma S_0} \int \frac{-dR}{(K_1 - R)^2 - \left( \sqrt{K_1^2 - K_2} \right)^2}, \\ t &= \frac{2\rho^2}{\gamma S_0} \times \frac{1}{2\sqrt{K_1^2 - K_2}} \log \frac{K_1 - R - \sqrt{K_1^2 - K_2}}{K_1 - R + \sqrt{K_1^2 - K_2}} + A \end{aligned}$$

$$t = \frac{\rho^2}{\gamma S_0 \sqrt{K_1^2 - K_2}} \log \frac{K_1 - R - \sqrt{K_1^2 - K_2}}{K_1 - R + \sqrt{K_1^2 - K_2}} + A,$$

where  $A$  is constant of integration.

When  $t = 0$ ,  $R = 0$ , therefore

$$A = -\log \frac{K_1 + \sqrt{K_1^2 - K_2}}{K_1 - \sqrt{K_1^2 - K_2}} = \alpha \text{ (let).}$$

Hence

$$\begin{aligned} t &= \frac{\rho^2}{\gamma S_0 \sqrt{K_1^2 - K_2}} \log \frac{K_1 - R - \sqrt{K_1^2 - K_2}}{K_1 - R + \sqrt{K_1^2 - K_2}} + \alpha \\ \Rightarrow \frac{(t - \alpha) \gamma S_0 \sqrt{K_1^2 - K_2}}{\rho^2} &= \log \frac{K_1 - R - \sqrt{K_1^2 - K_2}}{K_1 - R + \sqrt{K_1^2 - K_2}} \\ \Rightarrow \frac{K_1 - R - \sqrt{K_1^2 - K_2}}{K_1 - R + \sqrt{K_1^2 - K_2}} &= e^{\frac{(t - \alpha) \gamma S_0 \sqrt{K_1^2 - K_2}}{\rho^2}} = \beta, \end{aligned}$$

where  $\beta = e^{\frac{(t - \alpha) \gamma S_0 \sqrt{K_1^2 - K_2}}{\rho^2}}$ .

$$\begin{aligned} K_1 - R - \sqrt{K_1^2 - K_2} &= \beta K_1 - \beta R + \beta \sqrt{K_1^2 - K_2} \\ \Rightarrow R(\beta - 1) &= K_1(\beta - 1) + \sqrt{K_1^2 - K_2}(1 + \beta) \\ \Rightarrow R &= \frac{K_1(\beta - 1) + \sqrt{K_1^2 - K_2}(1 + \beta)}{\beta - 1} \\ \Rightarrow R &= K_1 + \sqrt{K_1^2 - K_2} \left( \frac{1 + \beta}{\beta - 1} \right). \end{aligned}$$

Hence

$$R = K_1 + \sqrt{K_1^2 - K_2} \left( \frac{\frac{(t-\alpha)\gamma S_0 \sqrt{K_1^2 - K_2}}{\rho^2}}{e \frac{(t-\alpha)\gamma S_0 \sqrt{K_1^2 - K_2}}{\rho^2} - 1} \right).$$

As  $t \rightarrow \infty$

$$R_\infty = K_1 + \sqrt{K_1^2 - K_2},$$

$$R_\infty = \rho \left( 1 - \frac{\rho}{S_0} \right) + \sqrt{\left[ \rho \left( 1 - \frac{\rho}{S_0} \right) \right]^2 - 2\rho^2 + \frac{2(S_0 + I_0)\rho^2}{S_0}},$$

$$R_\infty = \rho \left( 1 - \frac{\rho}{S_0} \right) + \rho \left( 1 - \frac{\rho}{S_0} \right), \quad \text{as } I_0 = 0.$$

Hence

$$R_\infty \approx 2\rho \left( 1 - \frac{\rho}{S_0} \right),$$

which is the steady state and this represents the size of propagation.

If  $S_0 < \rho$ , then from equation (2),  $\frac{dI}{dt}$  is initially negative and then virus propagation does not build up. Hence the virus propagation builds up only if  $S_0 > \rho$  or  $\rho < S_0$  i.e. only when the effective removal rate is less than the initial number of susceptible nodes and in this case all the nodes do not get infected. A stage may be reached when all the infected nodes are immediately removed. Thus a virus propagation builds up only when the density of susceptible is high owing to overcrowding and the rate of removal is low due to inadequate isolation facilities. On the other hand, if the isolation conditions are good and the density of susceptible is low, the virus propagation fades out.

Assume

$$S_0 = \rho + \nu.$$

For small  $\nu$  we can write

$$R_{\infty} \approx 2\rho \left(1 - \frac{\rho}{\rho + \nu}\right).$$

This implies  $R_{\infty} \approx 2\rho \left(\frac{\nu}{\rho + \nu}\right) \approx 2\nu$ ,

so that the initial density of susceptible  $\rho + \nu$  is reduced to a final density  $\rho - \nu$  i.e. the final density is as far below the threshold value  $\rho$  just as the initial density is above it.

Equation (2) shows that  $\frac{dI}{dt} < 0$  for  $S_0 < \rho$  and  $S(t) < S_0$ , so that the infection dies out and the virus propagation does not build up. Thus initial susceptible nodes must exceed a certain critical value for a virus propagation to grow. As  $S(t)$  is a monotonic decreasing function and is bounded below by  $S(t) \geq 0$ , so by Weierstrass's theorem,

$$\lim_{t \rightarrow \infty} R(t) = S(\infty).$$

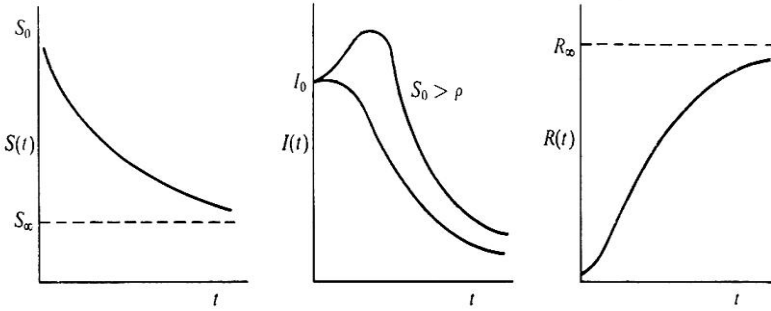
Again, from  $\frac{dR}{dt} = \gamma I$ ,  $R(t)$  is monotonic increasing function of  $t$  and is bounded above by  $R(t) \leq N$ ,

$$\lim_{t \rightarrow \infty} R(t) = R(\infty).$$

Thus,  $S(t) + I(t) + R(t) = N$  and  $\lim_{t \rightarrow \infty} I(t) = I(\infty)$ .

Thus,  $S(\infty)$ ,  $I(\infty)$  and  $R(\infty)$  determine how the virus propagation behaves, as  $\frac{dI}{dt} = \beta I(S - \rho)$  and if  $S_0 < \rho$ , then  $I(t)$  monotonically decreases to  $I(\infty)$ . If  $S_0 > \rho$ , then  $I(t)$  initially increases and continues to increase till  $S$  decreases to  $\rho$  and after that  $I(t)$  decreases to  $I(\infty)$ .

For  $S = S_0 e^{-R/\rho}$ ,  $t > 0$ ,  $R > 0$ , we get  $S > 0$ , then  $S(\infty) > 0$  so that there will always be susceptible nodes in the system and some nodes will always escape from infection. In particular, the spread of the virus will not stop for lack of susceptible nodes. The typical variation of  $S(t)$ ,  $I(t)$  and  $R(t)$  are shown below:



From Equation (1) and (2) we have

$$\frac{dI}{dS} = \frac{\beta I(S - \rho)}{-\beta SI},$$

$$\frac{dI}{dS} = -1 + \frac{\rho}{S},$$

$$\int dI = \int -dS + \int \frac{\rho}{S} dS,$$

$$I = -S + \rho \log S + A.$$

Applying the initial conditions;

$$I_0 = -S_0 + \rho \log S_0 + A,$$

$$A = I_0 + S_0 - \rho \log S_0,$$

we have

$$I = I_0 + S_0 - S + \rho \log \frac{S}{S_0},$$

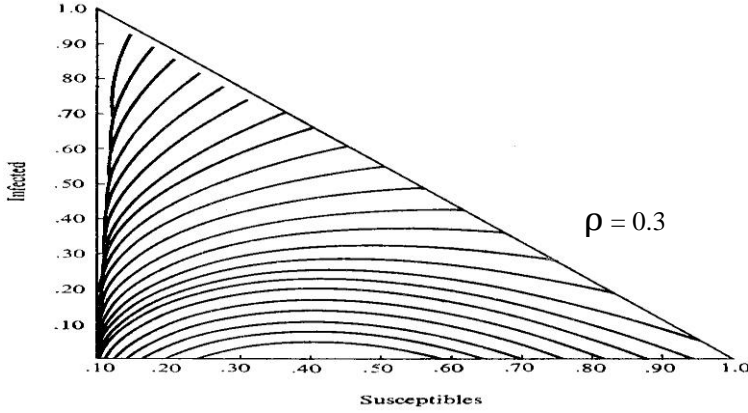
$$I = N - S + \rho \log \frac{S}{S_0},$$



or 
$$\frac{I}{N} = \frac{N}{N} - \frac{S}{N} + \frac{\rho}{N} \log \frac{S/N}{S_0/N},$$

$$\bar{I} = 1 - \bar{S} + \bar{\rho} \log \frac{\bar{S}}{\bar{S}_0}.$$

The graphs of the curves is



The curves are described from right to left and they continue to move to left till  $\frac{dS}{dt} = 0, I = 0$ , when  $\frac{dI}{dt}$  and  $\frac{dR}{dt}$  also become zero. Therefore  $I(\infty) = 0, S(\infty) + R(\infty) = N$ . The intensity of virus propagation is measured in terms of  $\frac{R(\infty)}{N}$ . The  $\frac{R(\infty)}{N}$  gives the proportion of nodes which contracts the virus.

The maximum number of infection takes place when  $\bar{S} = \bar{\rho}$  and it is independent of  $\bar{S}_0$ . If the initial point  $(\bar{S}_0, \bar{I}_0)$  is to the left of the line  $\bar{S} = \bar{\rho}$  the number of infective falls steadily to zero, if it is on the right of the line  $\bar{S} = \bar{\rho}$  the number of infective first increases and then decreases to zero. Since  $I_0 > 0$ , it can be seen that  $\bar{S}(\infty) < \bar{\rho}$  or  $S(\infty) < \rho$  and  $R(\infty) > N - \rho$ . Therefore,  $1 - \rho/N$  is less than the intensity of virus propagation.

## Conclusion

In this paper we have dealt with the realistic phenomena namely general deterministic model with removal of virus propagation in the system network. In this paper we have also discussed the asymptotic behaviour of the solution obtained under the deterministic model with removal and variations in different types of susceptible–infected and recovered nodes at any instant of time. The researcher can further enhance its variability by incorporating the effect of quarantine or application of anti malicious software.

## References

1. J. R. C. Piqueira, B. F. Navarro and H. A. Mouteiro, Epidemiological models applied to viruses in computer networks, *Journal of Computer Sciences*, **1(1)** (2005) 31-34.
2. B. K. Mishra and N. Jha, Fixed period of temporary immunity after run of anti-malicious software on computer models, *Applied Mathematics and Computation*, **190(2)** (2007) 1207-1212.
3. J. R. C. Piqueira and V. O. Araujo, A modified epidemiological model for computer viruses, *Applied Mathematics and Computation*, **213** (2009) 355-360.
4. X. Han and Q. Tan, Dynamical behavior of computer virus on internet, *Applied Mathematics and Computation*, **217(6)** (2010) 2520-2526.
5. J. O. Kephart and S. R. White, Directed-graph epidemiological models of computer viruses, *IEEE Symposium on Security and Privacy*, (1991) 343-361.
6. C. C. Zou, W. Gong and D. Towsley, Worm propagation modelling and analysis under dynamic quarantine defense, *Proceedings of the ACM CCS Workshop on Rapid Malcode (WORM'03)*, (2003) 51-60.
7. C. Suppo, J. M. Naulin, M. Langlais and M. Artois, A modelling approach to vaccination and contraception programmes for rabies control in fox populations, *Proc. R. Soc. Lond. B.*, **267(1452)** (2000) 1575-1582.