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On Transiso Graphs of Groups of Order Less Than 32

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Abstract: For a finite group G and a divisor d of $|G|$, the transiso graph $\Gamma_d(G)$ is a graph whose vertices are subgroups of G of order d and two distinct vertices H_1 and H_2 are adjacent if and only if there exist normalized right transversals S_1 and S_2 of H_1 and H_2 respectively in G such that $S_1 \cong S_2$ with respect to the right loop structure induced on them. In the present paper, we have determined some finite groups G for which the graphs $\Gamma_d(G)$ are complete for each divisor d of $|G|$. We have also discussed the completeness of transiso graphs for groups of order less than 32.

Keywords: Right loop, Normalized right transversal, Transiso graph, t-group.

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1. Introduction

Let G be a finite group and H be a subgroup of G. A normalized right transversal (NRT) S of H in G is a subset of G obtained by selecting one and only one element from each right coset of H in G and $1 \in S$. An NRT S has an induced binary operation $\circ$ given by $\{x \circ y\} = S \cap Hxy$, with respect to which S is a right loop with identity 1 (Smith[1], p.42, Lal[2]). Conversely, every right loop can be embedded as an NRT in a group with some universal property (Lal[2], p.76). Let $\langle S \rangle$ be the subgroup of G generated by S and H_S be the subgroup $H \cap \langle S \rangle$. Then, $H_S = \left\langle \left\{ xy(x \circ y)^{-1} \mid x, y \in S \right\} \right\rangle$ and $H_S S = \langle S \rangle$. Identifying S with the set $H \backslash G$ of all right cosets of H in G, we get a

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transitive permutation representation $\chi_S : G \to Sym(S)$ defined by $\{\chi_S(g)(x)\} = S \cap Hxg$, $g \in G$, $x \in S$. The kernel $\ker \chi_S$ of this action is $Core_G(H)$, the core of H in G. The group $G_S = \chi_S(H_S)$ is known as the group torsion of the right loop S (Lal[2], p. 75) which depends only on the right loop structure $\circ$ on S and not on the subgroup H. Since χ_S is injective on S and if we identify S with $\chi_S(S)$, then $\chi_S(\langle S \rangle) = G_S S$ which also depends only on the right loop S and S is an NRT of G_S in $G_S S$. One can also verify that $\ker\left(\chi_S |_{H_S S} : H_S S \to G_S S\right)$ $= \ker\left(\chi_S |_{H_S} : H_S \to G_S\right) = Core_{H_S S}(H_S)$ and $\chi_S |_S = I_S$, the identity map on S. If H is a corefree subgroup of G, then there exists an NRT T of H in G which generates G (Cameron[3]). In this case, $G = H_T T \cong G_T T$ and $H = H_T \cong G_T$. Also $(S, \circ)$ is a group if and only if G_S is trivial. Let $\mathcal{T}(G,H)$ denote the set of all normalized right transversals (NRTs) of H in G. Two NRTs $S, T \in \mathcal{T}(G,H)$ are said to be isomorphic (denoted by $S \cong T$), if their induced right loop structures are isomorphic. A subgroup H is normal in G if and only if all NRTs of H in G are isomorphic to the quotient group G/H (Lal[2]).

Throughout the paper, we will assume that G is a finite group and d is a divisor of the order $|G|$ of the group G. Let $V_d(G)$ be the set of all subgroups of G of order d. We define a graph $\Gamma_d(G) = (V_d(G), E_d(G))$ with $\{H_1, H_2\} \in E_d(G)$ if and only if there exists $S_i \in \mathcal{T}(G, H_i)$ $(i = 1, 2)$ such that $S_1 \cong S_2$ with respect to the right loop structure induced on S_i. We will call this graph a transiso graph (Kakkar and Mishra[4]). If G has no subgroup of order d, then $\Gamma_d(G)$ is a null graph (a graph having empty vertex set and empty edge set). If G has unique subgroup of order d, then $\Gamma_d(G)$ is an empty graph (a graph having empty edge set). We will denote transiso graph $\Gamma_d(G)$ by Γ_d if there is no confusion about G. A group G is called a t-group if $\Gamma_d(G)$ is a complete graph for each divisor d of $|G|$.

In this paper, we have determined all t-groups of the order less than 32. In the Section 2, we have recalled some preliminary results related to transiso graph from Kakkar and Mishra[4]. We have also discussed about the relation of adjacency and proved that the direct product of two t-groups of co-prime order is a t-group. In the Section 3, we have discussed about the transiso graphs of some non-abelian groups like dicyclic groups, quasidihedral groups and the groups of the order $pq, 4p, 2pq$ and $2p^2$ for distinct odd prime p and q. We have classified all the t-groups of order less than 32 in the Section 4.

2. Preliminaries

We first recall the following results of Kakkar and Mishra[4] and prove some elementary results which will be used in the present paper.

Proposition 1: *A subgroup of a group* G *is always adjacent with its automorphic images in* $\Gamma_d(G)$ *for any divisor* d *of* $|G|$.

Proposition 2: *Let* H_1 *and* H_2 *be corefree subgroups of* G. *Let* $S_i \in \mathcal{T}(G, H_i)$ $(i = 1, 2)$ *such that* $S_1 \cong S_2$ *and* $\langle S_i \rangle = G$. *Then, an isomorphism between* S_1 *and* S_2 *can be extended to an automorphism of* G *which sends* H_1 *onto* H_2.

Proposition 3: *A finite abelian group* G *is a t-group if and only if each Sylow subgroup of* G *is either elementary abelian or cyclic.*

Corollary 1: *An elementary abelian group is a t-group.*

Proposition 4: *The dihedral group* D_{2n} *of order* $2n$ *is a t-group.*

One can easily observe that the number of vertices in the graph is equal to the number of subgroups of order d and is given by

$$|V_d(D_{2n})| = \begin{cases} 1 & \text{if } d \text{ is odd.} \\ \dfrac{2n}{d} & \text{if } d \text{ is even and does not divide } n. \\ \dfrac{2n}{d} + 1 & \text{if } d \text{ is even and divides } n. \end{cases}$$

Proposition 5: *Let* G *be a non* p*-central finite* p*-group. Then,* $\Gamma_d(G)$ *is complete if and only if whenever* H *is a non-normal subgroup of* G *of order* p, $G \cong H \ltimes K$ *for some subgroup* K *of* G *with* $G/L \cong K$ *for any normal subgroup* L *of* G *of order* p.

Proposition 6: *Let p be an odd prime and G be a non-abelian group. Then,*

1. *If the group G is a t-group and $|G| = p^3$, then G is of exponent P (and hence $G \cong C_p{}^2 \rtimes C_p$).*
2. *If $|G| = p^4$, then $\Gamma_p(G)$ is not a complete graph.*
3. *If $|G| = p^5$, then $\Gamma_p(G)$ is not complete unless $\Phi(G) = Z(G) = G' \cong C_p^2$.*

Let G be a finite group and d be a divisor of $|G|$. Let us define a relation $\sim_d$ on the set $V_d(G)$ of all subgroups of the group G of order d such that two subgroups H_1 and H_2 are related by the relation $\sim_d$ if either $H_1 = H_2$ or H_1 and H_2 are adjacent in the graph $\Gamma_p(G)$. We call this relation $\sim_d$ the relation of adjacency in the graph $\Gamma_p(G)$. It is trivial that the relation $\sim_d$ is reflexive and symmetric on $V_d(G)$.

Proposition 2.1: *If the relation $\sim_d$ defined above is a transitive relation on $V_d(G)$, then $\Gamma_p(G)$ is either a complete graph or a disjoint union of complete graphs.*

Proof: Assume that the relation $\sim_d$ is a transitive relation on $V_d(G)$. Then, it is an equivalence relation on $V_d(G)$ and hence it gives a partition of $V_d(G)$ and each component of this partition corresponds to a complete graph.

Lemma 2.1: *Let H_i and K_i $(i = 1, 2)$ be subgroups of the groups G_i such that there exist NRTs $S_i \in \mathcal{T}(G, H_i)$ and $T_i \in \mathcal{T}(G, H_i)$ with $S_i \cong T_i$. Then , $S_1 \times S_2 \cong T_1 \times T_2$.*

Proof: One can easily observe that $S_1 \times S_2 \in \mathcal{T}\left(G_1 \times G_2, H_1 \times H_2\right)$, for an element $(g_1, g_2) \in G_1 \times G_2$ can be expressed as $(g_1, g_2) = (h_1 s_1, h_2 s_2)$ $= (h_1, h_2)(s_1, s_2)$, where $h_i \in H_i$ and $s_i \in S_i$ $(i = 1, 2)$. Similarly, $T_1 \times T_2 \in \mathcal{T}\left(G_1 \times G_2, K_1 \times K_2\right)$.. Then, the map $f \times g : S_1 \times S_2 \to T_1 \times T_2$ given by $(s_1, s_2) \in (f(s_1), g(s_2))$, is a right loop isomorphism where $f : S_1 \to T_1$ and $g : S_2 \to T_2$ are right loop isomorphisms.

Proposition 2.2: *The direct product of two t-groups of co-prime order is a t-group.*

Proof: Let G_1 and G_2 be two t-groups of co-prime order. Let $G = G_1 \times G_2$ and H, K be subgroups of G of same order. Then by [Suzuki[5], p. 141], $H = H_1 \times H_2$ and $K = K_1 \times K_2$ for some subgroups $H_1, K_1 \in G_1$ and $H_2, K_2 \in G_2$ such that $|H_1| = |K_1| = d_1$ and $|H_2| = |K_2| = d_2$. Since G_1 and G_2 are t-groups, $H_1 \sim_{d_1} K_1$ and $H_2 \sim_{d_2} K_2$. Therefore by Lemma 2.1, the subgroups H and K are adjacent in the corresponding transiso graph. Hence the group G is also a t-group.

Lemma 2.2: *Let G be a finite group and H be a non-normal subgroup of prime order. Then, an NRT S of H in G is either a subgroup of G or $H = H_S \cong G_S$.*

Proof: Let S be an NRT of H in G. Then, either $H_S = \{1\}$ or $H_S = H$. If $H_S = \{1\}$, then S is a subgroup of G. Now, assume that $H_S = H$. Since H is core-free, $G_S \cong H_S$. We also observe that S is not a group in this case.

3. Transiso Graphs for Some Non-Abelian Groups

In this section, we have determined transiso graphs for some non-abelian groups like dicyclic groups, quasidihedral groups and the groups of the order $pq, 4p, 2pq$ and $2p^2$ for distinct odd primes p and q. The dicyclic group (or binary dihedral group) $Q_{4n} = \langle a, b \mid a^{2n}, a^n b^2, abab^{-1} \rangle$ is a group of order $4n$ for $n \geq 1$ (Roman[6], p. 347). It is a non-abelian group for $n > 1$ and it is a cyclic group for $n = 1$ (that is, $Q_4 \cong C_4$). A generalized quaternion group is a special case of the dicyclic group Q_{4n} when $n = 2^k$ for some positive integer k.

In order to prove the Proposition 3.1, we need the following elementary lemma.

Lemma 3.1: *A subgroup of the dicyclic group Q_{4n} is either cyclic or dicyclic. Moreover, if d is a divisor of $4n$, then*

1. *there is unique subgroup (namely $\left\langle a^{\frac{2n}{d}}\right\rangle$) of Q_{4n} of order d if 4 does not divide d,*
2. *there are i subgroups $\left(\left\langle a^i, a^i b\right\rangle, 0 \le j < i\right)$ of order d conjugate to each other if 4 divides d and $i = \dfrac{4n}{d}$ is odd,*
3. *a subgroup of order d is either $\left\langle a^i\right\rangle$ or conjugate to one of $\left\langle a^i, b\right\rangle$ or $\left\langle a^i, ab\right\rangle$ if 4 divides d and $i = \dfrac{4n}{d}$ is even.*

Proof: Let H be a nontrivial proper subgroup of Q_{4n} of order d. Clearly $<a>$ is maximal cyclic subgroup of Q_{4n} of index 2. The composite homomorphism $H \to Q_{4n} \to Q_{4n}/<a>$ is either trivial or onto with the kernel $H \cap <a> = \left\langle a^i\right\rangle$ for unique divisor i of $2n$. If the homomorphism is trivial, then $H \cap <a> = \left\langle a^i\right\rangle$ for unique divisor $i = \dfrac{4n}{d}$ of $2n$. Therefore the subgroup H is cyclic in this case.

Now, if the homomorphism is onto, then $H/\left\langle a^i\right\rangle \cong Q_{4n}/<a> \cong C_2$. Since $H \not\subset <a>$, H has an element $a^j b$ and $a^n \subseteq \left\langle a^i\right\rangle$ for $\left(a^j b\right)^2 = a^n \in H$. Therefore $H \cap <a> = \left\langle a^i\right\rangle$ for unique divisor $i = \dfrac{4n}{d}$ of n. Now, we have an appropriate element $a^j b \in H \setminus <a>$ where $0 \le j < i$, such that $H = \left\langle a^i, a^j b\right\rangle$. Clearly H is a dicyclic group $\left(\text{precisely } \mathrm{H} \cong \mathrm{Q}_{4\frac{\mathrm{n}}{\mathrm{i}}}\right)$ for $\left(a^i\right)^{\frac{d}{2}} = 1$, $\left(a^i\right)^{\frac{d}{4}} = \left(a^j b\right)^2$ and $\left(a^j b\right) a^i \left(a^j b\right)^{-1} = \left(a^i\right)^{-1}$.

Now, we prove the next part of the lemma.

Let H be a subgroup of Q_{4n} of order d and $i = \dfrac{4n}{d}$. If d is not a multiple of 4, then there is no subgroup of Q_{4n} of order d which is dicyclic and so $H=\left\langle a^{\frac{i}{2}}\right\rangle$ is a cyclic subgroup. If d is a multiple of 4, then there are two cases.

If $d\nmid 2n$ i.e. i is odd, then H cannot be contained in $<a>$ so H is dicyclic subgroup of the form $\left\langle a^i, a^j b\right\rangle$. If $i\leq j$, then we can find l such that $0\leq l<i$ and $H=\left\langle a^i, a^l b\right\rangle$. Thus we conclude that $0\leq j<i$ and hence there are i subgroups of order d which are conjugates.

If $d\,|\,2n$ i.e., i is even, then H is either $\left\langle a^{\frac{i}{2}}\right\rangle$ or of the form $\left\langle a^i, a^j b\right\rangle$. Using above arguments, we see that there are $\frac{i}{2}$ subgroups conjugate to $\left\langle a^i, b\right\rangle$ and $\frac{i}{2}$ subgroups conjugate to $\left\langle a^i, ab\right\rangle$.

One can easily observe that an abelian normal subgroup of the group Q_{4n} is cyclic subgroup contained in the maximal cyclic subgroup and a non-abelian normal subgroup of Q_{4n} has index less than or equal to 2.

Proposition 3.1: *The dicyclic group* $Q_{4n}=\left\langle a,b\,|\,a^{2n}, a^n b^2, abab^{-1}\right\rangle$ *of order* $4n$ *is a t-group.*

Proof: Let d be a divisor of $4n$ and $i=\frac{4n}{d}$.

First assume that $4\nmid d$. Then by Lemma 3.1, there is unique subgroup of Q_{4n} of order d and so $\Gamma_d(Q_{4n})$ is trivially a complete graph.

Now assume that $4\,|\,d$ and i is odd. Then by Lemma 3.1, there are i subgroups of order d conjugate to $\left\langle a^i, b\right\rangle$ and so $\Gamma_d(Q_{4n})$ is a complete graph.

Finally assume that $4\,|\,d$ and i is even. Then, a subgroup of order d is either $H_1=\left\langle a^{\frac{i}{2}}\right\rangle$ or conjugate to exactly one of $H_2=\left\langle a^i, b\right\rangle$ or $H_3=\left\langle a^i, ab\right\rangle$. Note that H_1 is a normal subgroup of Q_{4n} and so its all NRTs are isomorphic to $Q_{4n}/H_1\left(\cong D_{2\frac{i}{2}}\right)$.

Now, choose $S_2 = \left\{ a^{2j+k} b^k \mid 0 \le j < \frac{i}{2}, k = 0,1 \right\}$ in $\mathcal{T}(Q_{4n}, H_2)$ and $S_3 = \left\{ a^{2j} b^k \mid 0 \le j < \frac{i}{2}, k = 0,1 \right\}$ in $\mathcal{T}(Q_{4n}, H_3)$. Note that $\langle S_2 \rangle = \langle a^2, ab \rangle$ and $\langle S_3 \rangle = \langle a^2, b \rangle$. Then, $H_{S_2} = \langle S_2 \rangle \cap H_2 = \langle a^i \rangle \trianglelefteq \langle S_2 \rangle$ and $H_{S_3} = \langle S_3 \rangle \cap H_3 = \langle a^i \rangle \trianglelefteq \langle S_3 \rangle$. Therefore $G_{S_2} = G_{S_3} = \{1\}$ and hence S_2 and S_3 are groups.

Let $\circ_2$ denote the induced binary operation on S_2 as described in the Section 1. One can observe that, $(a^2)^{\frac{i}{2}} = (ab)^2 = (ab \circ_2 a^2)^2 = 1$. This implies that $S_2 \cong D_{2\frac{i}{2}}$. One can similarly observe that $S_3 \cong D_{2\frac{i}{2}}$. This shows that the graph $\Gamma_d(Q_{4n})$ is complete.

It follows from the Lemma 3.1 that the number of vertices in the graph $\Gamma_d(Q_{4n})$ is given by

$$|V_d(Q_{4n})| = \begin{cases} 1 & \text{if } 4 \text{ does not divide } d. \\ \dfrac{4n}{d} & \text{if } 4 \text{ divides } d \text{ and } \dfrac{4n}{d} \text{ is odd}. \\ \dfrac{4n}{d} + 1 & \text{if } 4 \text{ divides } d \text{ and } \dfrac{4n}{d} \text{ is even}. \end{cases}$$

The quasidihedral (or semidihedral) group $QD_{2^n} = \left\langle a, b \mid a^{2^{n-1}}, b^2, baba^{2^{n-2}+1} \right\rangle$ is a non-abelian group of order 2^n where $n > 4$ (Gorenstein[7], p. 191). Its subgroup structure can be given by the following lemma.

Lemma 3.2: *A proper nontrivial subgroup of the quasidihedral group QD_{2^n} is either cyclic or dihedral or generalized quaternion.*

Proof: The proof is similar to that of the Lemma 3.1. From theorem 4.10 of Gorenstein[7] (p. 199), it follows that an abelian normal subgroup of the quasidihedral group QD_{2^n} of order $d = 2^m$ is cyclic (precisely $\left\langle a^{2^{n-m-1}} \right\rangle$) and a non-abelian normal subgroup of QD_{2^n} has index less than or equal to 2.

Now, we have the following proposition from which it follows that the quasidihedral group QD_{2^n} is not a t-group.

Proposition 3.2: *Let G be the quasidihedral group QD_{2^n} and $d=2^m$ be a divisor of 2^n. Then, the graph $\Gamma_d(G)$ is complete if and only if $d \neq 2$.*

Proof: First assume that $d \neq 2$. Then by Lemma 3.2, a subgroup of G of order $d=2^m$ is either $H_1=\left\langle a^{2^{n-m-1}}\right\rangle \cong C_{2^m}$ or conjugate to exactly one of $H_2=\left\langle a^{2^{n-m}}, b\right\rangle$ or $H_3=\left\langle a^{2^{n-m}}, ab\right\rangle$. Note that H_1 is a normal subgroup of QD_{2^n} and so its all NRTs are isomorphic to $QD_{2^n} \big/ H_1 \left(\cong D_{2^{n-m}}\right)$.
Now choose $S_2=\left\{a^{2j+k}b^k \mid 0 \leq j < 2^{n-m-1}, k=0,1\right\}$ in $\mathcal{T}\left(QD_{2^n}, H_2\right)$ and $S_3=\left\{a^{2j}b^k \mid 0 \leq j < 2^{n-m-1}, k=0,1\right\}$ in $\mathcal{T}\left(QD_{2^n}, H_2\right)$. Note that $\langle S_2\rangle = \langle a^2, ab\rangle$ and $\langle S_3\rangle = \langle a^2, b\rangle$. Then, $H_{S_2} = \langle S_2\rangle \cap H_2 = \left\langle a^{2^{n-m}}\right\rangle \trianglelefteq \langle S_2\rangle$ and $H_{S_3} = \langle S_3\rangle \cap H_3 = \left\langle a^{2^{n-m}}\right\rangle \trianglelefteq \langle S_3\rangle$. Therefore $G_{S_2} = G_{S_3} = \{1\}$ and hence S_2 and S_3 are groups.

Let $\circ_2$ denote the induced binary operation on S_2 as described in the Section 1. One can observe that, $\left(a^2\right)^{2^{n-m-1}} = (ab)^2 = \left(ab \circ_2 a^2\right)^2 = 1$. This implies that $S_2 \cong D_{2^{n-m}}$. One can similarly observe that $S_3 \cong D_{2^{n-m}}$. This shows that the graph $\Gamma_d\left(QD_{2^n}\right)$ is complete.
Finally assume that $d=2$. Then, a subgroup of G of order 2 is either $H_1=\left\langle a^{2^{n-2}}\right\rangle$ or a conjugate to $H_2=\langle b\rangle$. Since $H_1 \trianglelefteq G$, every NRT of H_1 in G is isomorphic to $G/H_1 \cong D_{2^{n-1}}$.

Let H be a non-normal subgroup of QD_{2^n} of order 2. Then, H is contained in $\langle a^2, b\rangle \cong D_{2^{n-1}}$ and H is a conjugate to the subgroup $\langle b\rangle$. Clearly the core $Core_G(H)$ of H in QD_{2^n} is trivial. Now let S be an NRT of H in QD_{2^n}. Then, the order of $H_S = H \cap \langle S\rangle$ is less than or equal to 2.

If $|H_S|=1$, then $S=\langle S\rangle$ is a subgroup of QD_{2^n}. Therefore S is equal to either $<a>$ or $\langle a^2,ab\rangle\cong Q_{2^{n-1}}$.

Finally if $|H_S|=2$, then $H_S=H$ and $\langle S\rangle=G$. Therefore, $G_S\cong H_S/Core_{H_S S}(H_S)=H/Core_G(H)\cong H$. Since G_S is nontrivial, S is not a group. Hence $S\ncong D_{2^{n-1}}$.

It can be trivially observed that the number of vertices in the graph $\Gamma_d(QD_{2^n})$ is equal to the number of subgroups of QD_{2^n} of order d and is given by

$$\left|V_d\left(QD_{2^n}\right)\right|=\begin{cases}1 & if\ d=1\ or\ d=2^n.\\ 2^{n-2}+1 & if\ d=2.\\ 2^{n-m}+1 & if\ d=2^m\ with\ 0<m<n.\end{cases}$$

Proposition 3.3: *Let p and q be distinct odd primes. Then, a group of order either pq or $4p$ or $2pq$ is t-group.*

Proof: Observe that a nontrivial proper subgroup of a group of order pq is a Sylow subgroup. Hence any two subgroups of same order are adjacent in corresponding transiso graph.

By classification of groups of order $4p$ (Burnside[8], p.132-137), a non-abelian group of order $4p$ is isomorphic to exactly one of D_{4n}, Q_{4n}, the alternating group $Alt(4)$ (for $p=3$), $C_p\rtimes C_4$ (for $p\equiv 1 mod\, 4$). The groups D_{4n} and Q_{4n} are t-groups from the propositions 4 and 3.1. Since any two subgroups of the group $Alt(4)$ of equal order are conjugate therefore the group $Alt(4)$ is also a t-group.

Let H_1 and H_2 be two distinct subgroups of $C_p\rtimes C_4$ of order 2. Then, there exist unique Sylow 2-subgroup K_i of $C_p\rtimes C_4$ containing H_i where $i=1,2$. Since K_1 and K_2 are conjugate, the subgroups H_1 and H_2 are conjugate. So H_1 and H_2 are adjacent in $\Gamma_2(C_p\rtimes C_4)$.

A non-abelian group of order $2pq$ is isomorphic to exactly one of the groups D_{2pq}, $D_{2q}\times C_p$, $D_{2p}\times C_q$ and $C_2\times(C_q\rtimes C_p)$, $(C_q\rtimes C_p)\rtimes C_2$ (when

p divides $q-1$) (Ghorbani and Larki[9], p. 50). $D_{2q}\times C_p$, $D_{2p}\times C_q$ and $C_2\times(C_q\rtimes C_p)$ are t-groups due to the Proposition 2.2. Order of the normalizer $N_G(H)$ of a Sylow p-subgroup H of $(C_q\rtimes C_p)\rtimes C_2$ is $2p$ and H is unique Sylow p-subgroup of $N_G(H)$. Since all Sylow p-subgroups are conjugate; therefore their normalizers are also conjugate.

Proposition 3.4: *Let G be a non-abelian group of order $2p^2$ for some odd prime p. Then, the group G is t-group if and only if G is isomorphic to either the dihedral group D_{2p^2} or $(C_p)^2\rtimes C_2$.*

Proof: It is well known that a non-abelian group of order $2p^2$ is isomorphic to exactly one of the groups D_{2p^2}, $(C_p)^2\rtimes C_2$ and $C_p\times D_{2p}$ (Burnside[8], p.132-137).

Let $G=\langle a,b,c\,|\,a^p,b^p,c^2,[a,b],(ac)^2,(bc)^2\rangle\cong\left(C_p\right)^2\rtimes C_2$. Then, all subgroups of $\langle a,b\rangle\cong\left(C_p\right)^2$ are normal in G and their quotients are dihedral groups D_{2p}. Hence $\Gamma_p(G)$ is a complete graph. Now $\Gamma_{2p}(G)$ is also complete as there are several NRTs of a subgroup H of G order $2p$ which are isomorphic to the cyclic group of order p. So G is a t-group.

Now, let $G\cong C_p\times D_{2p}=\langle a,b,c\,|\,a^p,b^p,c^2,[a,b],[a,c],(bc)^2\rangle$. Then, it is obvious that $<a>$ and $<b>$ are normal subgroups of G of order p such that $G/<a>\cong D_{2p}$ and $G/<b>\cong C_{2p}$. Hence $\Gamma_p(G)$ is not a complete graph.

4. Classification of T-Groups of Order Less Than 32

Abelian t-groups are already determined by Proposition 3 which tells that a finite abelian group G is a t-group if and only if it is isomorphic to the direct sum of a cyclic group C and a direct sum A of some elementary abelian groups, where $|A|$ and $|C|$ are co-prime.

Non-abelian groups of the order 12, 20, 21, 28 and 30 are t-groups by Proposition 3.3 and a non-abelian t-group of the order 18 can be determined by Proposition 3.4. By Propositions 3.1 and 4, it is clear that the non-abelian groups of order 8 and $2p$ (for odd prime $p \le 13$) are t-groups. In Propositions 4.1 and 4.2, we have determined non-abelian t-groups of the order 16 and 24 respectively. We recall that a finite p-group P is p-central if each subgroup of P of order p is contained in the center $Z(P)$.

Proposition 4.1: *Let G be a non-abelian group of order 16. Then, the group G is a t-group if and only if G is isomorphic to either dihedral group D_{16} or dicyclic group Q_{16}.*

Proof: If G is a 2-central group, then it is isomorphic to one of the groups $Q_8, C_4 \rtimes C_4$ and $C_2 \times Q_8$ (Wild[10]). By Proposition 3.1, Q_{16} is a t-group. The group $C_4 \rtimes C_4 = \langle a, b \mid a^4, b^4, abab^{-1} \rangle$ has three normal subgroups $\langle a^2 \rangle, \langle b^2 \rangle$ and $\langle a^2 b^2 \rangle$ of order 2 with quotient groups isomorphic to the groups $C_4 \times C_2$, D_8 and Q_8 respectively. Therefore the graph $\Gamma_2(C_4 \rtimes C_4)$ is not complete and hence $C_4 \rtimes C_4$ is not a t-group. The group $C_2 \times Q_8 = \langle a, b, c \mid a^2, b^4, b^2c^2, [a,b], [a,c], bcbc^{-1} \rangle$ is not a t-group, for it has three normal subgroups $\langle a \rangle, \langle b^2 \rangle$ and $\langle ab^2 \rangle$ of order 2 with quotient groups isomorphic to the groups Q_8, $(C_2)^3$ and Q_8 respectively. Therefore $C_4 \rtimes C_4$ is not a t-group.

If G is a non 2-central group which is also a t-group, then $\Gamma_2(G)$ is a complete graph and hence by Proposition 5, G should be isomorphic to a nontrivial semidirect product $H \ltimes K$ of a non-normal subgroup H of G of order 2 and a normal subgroup K of G of order 8 such that for any normal subgroup L of G of order 2, K is isomorphic to G/L. From a result of Wild[10] we observe that there are five groups $(C_4 \times C_2) \rtimes_1 C_2$, $C_8 \rtimes C_2$, $QD_{16} = C_8 \rtimes_1 C_2$, $D_{16} = D_8 \rtimes C_2$ and $(C_4 \times C_2) \rtimes_2 C_2$ of required semidirect product type. Proposition 4 asserts that the group D_{16} is a t-group and the group QD_{16} is not a t-group by Proposition 3.2. The groups $(C_4 \times C_2) \rtimes_1 C_2$, $C_8 \rtimes C_2$ and $(C_4 \times C_2) \rtimes_2 C_2$ have normal subgroups of order 2 such that corresponding quotient groups are isomorphic to D_8, $C_4 \times C_2$ and $(C_2)^3$ respectively[10]. Therefore these groups are not t-groups.

Lemma 4.1: *Let* G *be the group* $C_2 \times Alt(4)$. *Then, the graph* $\Gamma_2(G)$ *is not a complete graph.*

Proof: First note that $N = C_2 \times \{1\}$ is a normal subgroup of $G = C_2 \times Alt(4)$ of order 2, where I is the identity element of $Alt(4)$ and every NRT of N in G is isomorphic to $G/N \cong Alt(4)$.

Now, choose a non-normal subgroup H of G of order 2 which is contained the subgroup $C_2 \times Alt(4)$ of G.

Let S be an NRT of H in G. Note that $S' = S \cap (C_2 \times Alt(4))$ is an NRT of H in $C_2 \times Alt(4)$ and $\langle S' \rangle = C_2 \times Alt(4)$. Hence by Lemma 2.2, S can not be a group. Thus, the subgroups H and N are not adjacent in the graph $\Gamma_2(G)$, that is, the graph $\Gamma_2(G)$ is not complete.

Proposition 4.2: *Let* G *be a non-abelian group of order* 24. *Then, the group* G *is a t-group if and only if* G *is isomorphic to a semidirect product of two t-groups of co-prime order except the groups* $C_2 \times Alt(4)$ *and* $(C_2 \times C_6) \rtimes C_2$.

Proof: We know that there are 12 non isomorphic non-abelian groups of order 24 (Burnside[8], p.101-104) and 9 of them are semidirect product of two t-groups of co-prime order.

It is obvious that the groups $C_3 \rtimes C_8$ and $SL(2,3)$ are t-groups, for any two subgroups of respective groups of equal order are conjugate. The groups Q_{24} and D_{24} are also t-groups by Propositions 3.1 and 4 respectively. By Proposition 2.2, we see that the groups $C_3 \times D_8$, $C_3 \times Q_8$ and $C_2 \times D_{12} \cong (C_2)^2 \times D_6$ are t-groups. It is clear from example 2.2 of Kakkar and Mishra[4] that the symmetric group $Sym(4)$ is not a t-group. One can observe that $\langle a^2 \rangle \times Alt(3) \cong C_6$ and $\{1\} \times Sym(3)$ are normal subgroups of the group $< a > \times Sym(3) \cong C_4 \times D_6$ such that their quotient groups are $(C_2)^2$ and C_4 respectively. So $\Gamma_6(C_4 \times D_6)$ is not a complete graph and hence the group $C_4 \times D_6$ is not a t-group. Similarly $C_2 \times D_{12}$ is not a t-group since there are two normal subgroups $C_2 \times \{1\}$ and $\{1\} \times Z(D_{12})$ of order 2 such that their quotient groups are D_{12} and Q_{12}.

Now, consider $G=(C_2 \times C_6)\rtimes C_2$. It has a normal subgroup H of order 2 such that $G/H \cong D_{12}$. Let K be a subgroup of G of order 2 contained in the subgroup isomorphic to D_{12}. Then, there is no NRT $S \in \mathcal{T}(G,H)$ such that $S = D_{12}$, for otherwise $S=\langle S\rangle$ and $S \cap H = H$ which contradicts the fact that S is an NRT. Therefore the group $(C_2 \times C_6)\rtimes C_2$ is not a t-group. Finally by Lemma 2.2, the group $C_2 \times Alt(4)$ is not a t-group.

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