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Exterior Calculus Perspective of Maxwell's Electromagnetic Field Equations

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Abstract: The classical Maxwell's electromagnetic equations were obtained for electric field and magnetic field in terms of differential operators curl and divergence; these involve space coordinates and their directions, but use time as a parameter only. In Special Relativity time has been given the status of a coordinate and a direction is assigned. Using differential forms we have obtained in an entirely new way expressions for these differential operators which are invariant under Lorentz transformation: LT. Using these new derivations we obtain expressions for curl and divergence of electric field and magnetic field. These expressions are related to classical Maxwell's equations in such a natural way that they lead again to the conclusion that the latter are invariant under LT and provide an Exterior Calculus Perspective of Maxwell's equations and their presentation from the point of view of 4-dimensional space time.

Keywords: Lorentz transformation, Invariance of differential operators and Maxwell's equations.

2010AMS Classification No.: 83A05.

1. Introduction

In an earlier paper published as part of a book[1], we studied a LT where the moving frame moved with uniform velocity v in an arbitrary direction relative to the stationary frame. We obtained some new results which were invariant under LT. These helped us to obtain expressions for gradient, divergence, curl and d'Alembertian operator which are invariant under LT. The computations involved in obtaining these new results are quite heavy. In fact, they are not easily accessible to readers unfamiliar with the theory of differential forms in Exterior Calculus. To make our new results more easily understandable we used standard LT and obtained the same new results as in our earlier paper. These results are available in Krisna S. Amur and Christopher R[2]. Electromagnetic equations were written down, but at that time a suitable form needed to relate them to classical Maxwell's equations did not show itself up. This work is taken up in this paper.

We give some mathematical details which will help the reader to understand the expressions obtained for differential operators: gradient, divergence, curl and d'Alembertian.

Consider a coordinate system (x^0, x^1, x^2, x^3) in 4-dimensional space time with $x^0 = ct$. We assume that the velocity of light c is unity, so that $x^0 = t$ has the unit of length and x^1, x^2, x^3 are space coordinates. This space time has a flat metric η which satisfies

(1.1) $$\begin{aligned} \eta(e_\alpha, e_\beta) = \eta_{\alpha\beta} &= -1 \quad \text{if } \alpha = \beta = 0 \\ &= 1 \quad \text{if } \alpha, \beta = 1,2,3 \\ &= 0 \quad \text{if } \alpha \neq \beta \end{aligned}$$

where e_α , $\alpha = 0,1,2,3$ are basis vectors. This space time with metric η is known as Minkowski space.

Standard LT is a transformation from one system of coordinates x^α in a Lorentz frame Σ into another system $x^{\beta'}$ in a Lorentz frame Σ' which is moving with uniform velocity v along x^1-axis.The transformation equations are given by[3]

(1.2) $$x^{\beta'} = \wedge^{\beta'}_{\alpha} x^\alpha \quad , \quad x^\alpha = \wedge^{\alpha}_{\beta'} x^{\beta'},$$

where we have followed Einstien summation convention for repeated indices and the coefficient matrices are constant matrices given by

$$\left\| \wedge^{\alpha'}_{\beta} \right\| = \left\| \begin{matrix} \Gamma & -\Gamma v & 0 & 0 \\ -\Gamma v & \Gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \right\|, \qquad \left\| \wedge^{\alpha}_{\beta'} \right\| = \left\| \begin{matrix} \Gamma & \Gamma v & 0 & 0 \\ \Gamma v & \Gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \right\|,$$

where $\Gamma = \dfrac{1}{\sqrt{1-v^2}}$. Using these we have

$$x^{0'} = \Gamma x^0 - \Gamma v x^1\ ,\ x^{1'} = -\Gamma v x^0 + \Gamma x^1,\quad x^{2'} = x^2,\ x^{3'} = x^3$$
$$e_{0'} = \Gamma e_0 + \Gamma v e_1\ ,\quad e_{1'} = \Gamma v e_0 + \Gamma e_1,\ e_{2'} = e_2\ ,\ e_{3'} = e_3 .$$

We have

$$\boldsymbol{X} = \left(x^0, x^1, x^2, x^3\right) = x^\alpha e_\alpha, \quad \boldsymbol{X}' = \left(x^{0'}, x^{1'}, x^{2'}, x^{3'}\right) = x^{\beta'} e_{\beta'} .$$

The volume element dV and star operator $*dX$ in space time are given by

$$dV = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^0\ ,$$

where $\wedge$ is wedge product, it is skew symmetric, e.g $dx^1 \wedge dx^2 = -dx^2 \wedge dx^1$.

$$\begin{aligned} *d\boldsymbol{X} &= dx^2 \wedge dx^3 \wedge dx^0 e_1 + dx^3 \wedge dx^1 \wedge dx^0 e_2 \\ &\quad + dx^1 \wedge dx^2 \wedge dx^0 e_3 + dx^1 \wedge dx^2 \wedge dx^3 e_0\ . \end{aligned}$$

Lemma 1.1: *Both dV and *d**X** are invariant under LT.*

Proof: For details of the proof we refer[1,2].

Lemma 1.2: *Let f be differentiable real valued function and **F**, a differentiable vector field on space time. Then we have*

$$(1.3)\quad \begin{cases} (a) & (\mathrm{grad} f\,)_4 = \left(-\dfrac{\partial f}{\partial x^0}e_0 + \dfrac{\partial f}{\partial x^1}e_1 + \dfrac{\partial f}{\partial x^2}e_2 + \dfrac{\partial f}{\partial x^3}e_3\right), \\ (b) & \left(div\boldsymbol{F}\right)_4 = \left(\dfrac{\partial F^0}{\partial x^0} + \dfrac{\partial F^1}{\partial x^1} + \dfrac{\partial F^2}{\partial x^2} + \dfrac{\partial F^3}{\partial x^3}\right), \\ (c) & (curl\boldsymbol{F}\,)_4 = \left(gradF^0 + \dfrac{\partial \overline{\boldsymbol{F}}}{\partial x^0}\right) \times e_0 + (curl\overline{\boldsymbol{F}}\,)_3\ , \\ & \quad where\ \overline{\boldsymbol{F}} = \sum_{i=1}^{3} F^i e_i\ is\ spatial\ part\ of\ F, \\ (d) & d'Alembertian\ of\ f = \left(-\dfrac{\partial^2 f}{\partial x^{0^2}} + \dfrac{\partial^2 f}{\partial x^{1^2}} + \dfrac{\partial^2 f}{\partial x^{2^2}} + \dfrac{\partial^2 f}{\partial x^{3^2}}\right), \\ & where\ x^0 = ct. \end{cases}$$

These are invariant under LT.

Proof: We give a detailed proof of b) and for the proof of the rest we refer[1,2].
Consider a differentiable vector field $\boldsymbol{F} = F^{\alpha} e_{\alpha}$ on space time. Then setting $\boldsymbol{X} = x^{\mu} e_{\mu}$ we have

$$d\boldsymbol{F} \cdot \wedge * \, d\boldsymbol{X} = \frac{\partial F^{\alpha}}{\partial x^{\beta}} dx^{\beta} \wedge * dx^{\mu} < e_{\alpha}, e_{\mu} >, \quad \alpha, \beta, \mu = 0, 1, 2, 3$$

$$= \frac{\partial F^{\alpha}}{\partial x^{\beta}} \eta_{\alpha\mu} \eta^{\beta\mu} dV \,,$$

where $< e_{\alpha}, e_{\mu} > = \eta_{\alpha\mu}$ and $dx^{\beta} \wedge * dx^{\mu} = \eta^{\beta\mu} dV$

$$= \frac{\partial F^{\alpha}}{\partial x^{\beta}} \delta^{\beta}_{\alpha} dV \text{, on simplification,}$$

$$= \frac{\partial F^{\alpha}}{\partial x^{\alpha}} dV = \left(\frac{\partial F^{0}}{\partial x^{0}} + \frac{\partial F^{1}}{\partial x^{1}} + \frac{\partial F^{2}}{\partial x^{2}} + \frac{\partial F^{3}}{\partial x^{3}} \right) dV$$

$$= (div \boldsymbol{F})_4 dV \text{, by definition of divergence.}$$

Similarly, we can obtain

$$d\boldsymbol{F} \cdot \wedge * \, d\boldsymbol{X}' = (div \boldsymbol{F})'_4 dV' \,,$$

where $$(\mathrm{div} \boldsymbol{F})'_4 = \left(\frac{\partial F^{0'}}{\partial x^{0'}} + \frac{\partial F^{1'}}{\partial x^{1'}} + \frac{\partial F^{2'}}{\partial x^{2'}} + \frac{\partial F^{3'}}{\partial x^{3'}} \right).$$

Since $* d\boldsymbol{X} = * d\boldsymbol{X}'$, $dV = dV'$, it follows that $(div \boldsymbol{F})_4 = (div \boldsymbol{F})'_4$, hence $(div \boldsymbol{F})_4$ is invariant under LT.

2. Maxwell's Equations

In 3-dimensional Euclidean space in which charge density $\rho = 0$ and electric current density $\boldsymbol{J} = 0$ Maxwell's equations are given by[4]

$$(2.1)\qquad \begin{cases} (a) & (\mathrm{curl}\boldsymbol{E})_3 = -\dfrac{1}{c}\dfrac{\partial \boldsymbol{H}}{\partial t}, \\ (b) & (\mathrm{div}\boldsymbol{E})_3 = 0, \\ (c) & (\mathbf{curl}\boldsymbol{H})_3 = \dfrac{1}{c}\dfrac{\partial \boldsymbol{E}}{\partial t}, \\ (d) & (\mathrm{div}\boldsymbol{H})_3 = 0. \end{cases},$$

Both electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{H}$ have no component in the direction e_0 of time, we set $E^0 = 0$ and $H^0 = 0$, $\boldsymbol{E} = \overline{\boldsymbol{E}}$ and $\boldsymbol{H} = \overline{\boldsymbol{H}}$, where $\overline{\boldsymbol{E}}$ and $\overline{\boldsymbol{H}}$ are spatial parts of $\boldsymbol{E}$ and $\boldsymbol{H}$ respectively. Consider the divergence of these fields. From (1.3)(b) we have

$$(\mathrm{div}\boldsymbol{E})_4 = \sum_{i=1}^{3}\frac{\partial E^i}{\partial x^i} = (\mathrm{div}\boldsymbol{E})_3 \ , \quad \text{since } E^0 = 0.$$

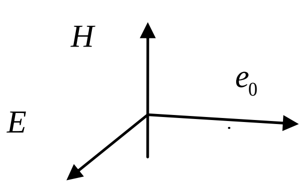
Similarly we can show that, $(\mathrm{div}\boldsymbol{H})_4 = (\mathrm{div}\boldsymbol{H})_3$.
Using the results in (2.1) we get

$$(2.2)\qquad (\mathrm{div}\boldsymbol{E})_4 = 0 \quad \text{and} \quad (\mathrm{div}\boldsymbol{H})_4 = 0.$$

Now consider curl of these fields. Using the formula for curl given in (1.3) c) we get

$$(2.3)\qquad \begin{cases} (\mathrm{curl}\boldsymbol{E})_4 = \dfrac{\partial}{\partial x^0}(\boldsymbol{E}\times e_0) + (\mathrm{curl}\boldsymbol{E})_3, \\ (\mathbf{grad}E^0)_4 = 0, \\ (\mathrm{curl}\boldsymbol{H})_4 = \dfrac{\partial}{\partial x^0}(\boldsymbol{H}\times e_0) + (\mathrm{curl}\boldsymbol{H})_3, \\ (\mathbf{grad}H^0)_4 = 0. \end{cases}$$

We observe that e_0 is orthogonal to both $\boldsymbol{E}$ and $\boldsymbol{H}$ which in their turn are orthogonal to each other.



So we have $\boldsymbol{E} \times e_0 = \boldsymbol{H}$ and $\boldsymbol{H} \times e_0 = -\boldsymbol{E}$. Using these in (2.3) we get

$$\text{(2.4)} \qquad \begin{cases} (\mathrm{curl}\boldsymbol{E})_4 = \dfrac{\partial \boldsymbol{H}}{\partial x^0} + (\mathrm{curl}\boldsymbol{E})_3 \,, \\ (\mathbf{curl}\boldsymbol{H})_4 = -\dfrac{\partial \boldsymbol{E}}{\partial x^0} + (\mathbf{curl}\boldsymbol{H})_3, \end{cases}$$

where $x^0 = ct$.

Using these and (2.1), (2.2) we state the following proposition.

Proposition 2.1: *In 4-dimensional space time which is free of charge density* ρ *and electric current density* $\boldsymbol{J}$ *, Maxwell's equations are given by*

$$\textit{(2.5)} \qquad \begin{cases} (\mathrm{div}\boldsymbol{E})_4 = 0, \quad (\mathrm{div}\boldsymbol{H})_4 = 0, \\ (\mathrm{curl}\boldsymbol{E})_4 = \dfrac{1}{c}\dfrac{\partial \boldsymbol{H}}{\partial t} + (\mathrm{curl}\boldsymbol{E})_3 = 0 \,, \\ (\mathrm{curl}\boldsymbol{H})_4 = -\dfrac{1}{c}\dfrac{\partial \boldsymbol{E}}{\partial t} + (\mathrm{curl}\boldsymbol{H})_3 = 0. \end{cases}$$

Since the expression for curl and divergence are invariant under LT, it follows that the classical Maxwell's electromagnetic equations expressed in terms of them are also invariant under LT.

We consider an application of the results obtained in Proposition 2.1. In[2] it is shown that $\left[\mathrm{curl}\left(\mathrm{grad}f\right)\right]_4 = 0$, where f is a real valued function on space time. Since $\left(\mathrm{curl}\boldsymbol{E}\right)_4 = 0$, we set $\boldsymbol{E} = \left(\mathrm{grad}f\right)_4$. Then since $\left(\mathrm{div}\boldsymbol{E}\right)_4 = 0$, we get $\left[\mathrm{div}\left(\mathrm{grad}f\right)\right]_4 =$ d'Alembertian of f =0. This leads to the wave equation[5]

$$\text{(2.6)} \qquad \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^{1\,2}} + \frac{\partial^2 f}{\partial x^{2\,2}} + \frac{\partial^2 f}{\partial x^{3\,2}},$$

from the formula (d) given in (1.3).

Proposition 2.2: *General classical Maxwell's equations can be put in the following form:*

$$(2.7)\qquad \begin{cases} (a) & (\mathrm{Curl}\boldsymbol{E})_4 = \dfrac{1}{c}\dfrac{\partial}{\partial t}(\boldsymbol{H}-\boldsymbol{B}), \\ (b) & (\mathrm{Curl}\boldsymbol{H})_4 = \dfrac{1}{c}\dfrac{\partial}{\partial t}(\boldsymbol{D}-\boldsymbol{E}) + \dfrac{4\pi}{c}\boldsymbol{J}\ , \\ (c) & (\mathrm{div}\boldsymbol{D})_4 = 4\pi\rho, \\ (d) & (\mathrm{div}\boldsymbol{B})_4 = 0, \end{cases}$$

where

B=*magnetic induction,* ***D***=*dielectric displacement,*
ρ=*charge density,* ***J***=*electric current density.*

Proof: General classical Maxwell's equations are[4]

$$(2.8)\qquad \begin{cases} (a) & (curl\boldsymbol{E})_3 = -\dfrac{1}{c}\dfrac{\partial \boldsymbol{B}}{\partial t}\ , \\ (b) & (curl\boldsymbol{H})_3 = \dfrac{4\pi}{c}\boldsymbol{J} + \dfrac{1}{c}\dfrac{\partial \boldsymbol{D}}{\partial t}\ , \\ (c) & (div\boldsymbol{D})_3 = 4\pi\rho, \\ (d) & (div\boldsymbol{B})_3 = 0. \end{cases}$$

From (2.4) and (2.8) we have

$(\mathrm{Curl}\boldsymbol{E})_4 = \dfrac{1}{c}\dfrac{\partial \boldsymbol{H}}{\partial t} + (\mathrm{Curl}\boldsymbol{E})_3 = \dfrac{1}{c}\dfrac{\partial}{\partial t}(\boldsymbol{H}-\boldsymbol{B})$ by using (2.8)(a), $x^0=ct$.

$(\mathrm{Curl}\boldsymbol{H})_4 = -\dfrac{1}{c}\dfrac{\partial \boldsymbol{E}}{\partial t} + (\mathrm{Curl}\boldsymbol{H})_3 = \dfrac{1}{c}\dfrac{\partial}{\partial t}(\boldsymbol{D}-\boldsymbol{E}) + \dfrac{4\pi}{c}\boldsymbol{J}$ by using (2.8)(b).

Since B^0 and D^0 are zero, $\dfrac{\partial B^0}{\partial t}$ and $\dfrac{\partial D^0}{\partial t}$ are zero in the expressions for $(div\boldsymbol{B})_4$ and $(div\boldsymbol{D})_4$ and $(\mathrm{div}\boldsymbol{B})_4 = (\mathrm{div}\boldsymbol{B})_3 = 0$ and $(\mathrm{div}\boldsymbol{D})_4 = (\mathrm{div}\boldsymbol{D})_3 = 4\pi\rho$.

Since classical Maxwell's electrodynamic equations are shown to be invariant under LT, it follows that the equations in (2.7) are invariant under LT.

Remark: The advantage of finding an expression for $(\mathrm{curl}\boldsymbol{F})_4$ which contains $\mathrm{curl}\overline{\boldsymbol{F}}$ where $\overline{\boldsymbol{F}}$ is spatial part of $\boldsymbol{F}$ is clearly seen in the computations given above. Classical Maxwell's equations in free space time show themselves in the expressions for $(\mathrm{curl}\boldsymbol{E})_4$ and $(\mathrm{curl}\boldsymbol{H})_4$.

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