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A Five-Dimensional C^h-Symmetric Finsler Space with Constant Unified Main Scalar*

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Abstract: In the present paper, we have discussed the theory of a five-dimensional C^h-symmetric Finsler space with constant unified main scalar. Also, the h-connection vectors of a five-dimensional C^h-symmetric Finsler space with constant unified main scalar has been determined.

Keywords: C^h-symmetric Finsler space, Miron frame unified main scalar, Landsberg space.

AMS Classification No. : 53B40, 53C60.

1. Introduction

F. Ikeda[1] discussed the properties of Finsler spaces satisfying the condition $L^2C^2 = f(x)$, where L is the fundamental function and C is the length of the torsion vector C_i. Also, Ikeda[2] considered the condition $L^2C^2 =$ non-zero constant, which is stronger than the corresponding condition considered in 1984. A two-dimensional Berwald space is an example of such a Finsler space with constant function LC. A theory of intrinsic orthonormal frame field on an n-dimensional Finsler space, as a generalization of Berwald's and Moor's ideas on two-dimensional and three-dimensional Finsler spaces respectively, has been studied by Matsumoto and Miron[3].

A three-dimensional Finsler space with constant unified main scalar has been studied by Ikela[4] and Singh and Kumari[5]. The four-dimensional C^h – symmetric Finsler space with constant unified main scalar has been studied by Tiwari and Rai[6].

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The main scalar of a five-dimensional Finsler space has been studied by Shanker, Chaubey and Pandey[7]. A Finsler space F^n is called C^h-symmetric Finsler space if $C_{ijk|h} = C_{ijh|k}$. In the present paper, we have discussed the theory of a five-dimensional C^h-symmetric Finsler space with constant unified main scalar. The orthonormal frame field $\left(l^i, m^i, n^i, p^i, q^i\right)$, called the Miron frame, plays an important role in a five-dimensional C^h-symmetric Finsler space.

2. Scalar Components in Miron Frame

Let us consider a five-dimensional Finsler space F^5 with the fundamental function $L(x, y)$. The metric tensor g_{ij} and $C-$tensor C_{ijk} of F^5 are defined by

$$g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2,\; C_{ijk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k L^2.$$

Throughout this paper, we use the symbols $\dot{\partial}_i = \dfrac{\partial}{\partial y^i}$ and $\partial_i = \dfrac{\partial}{\partial x^i}$. The frame $\left\{e^i_{(\alpha)}\right\}$, $\alpha = 1, 2, 3, 4, 5$ is called Miron frame of F^5, where $e^i_{(1)} = l^i = \dfrac{y^i}{L}$ is the normalized supporting element, $e^i_{(2)} = m^i = \dfrac{c^i}{c}$ is the normalized torsion vector, $e^i_{(3)} = n^i$, $e^i_{(4)} = p^i$, $e^i_{(5)} = q^i$ are constructed by $g_{ij}e^i_{(\alpha)}e^i_{(\beta)} = \delta_{\alpha\beta}$. Here C is the length of the torsion vector $C_i = C_{ijk}g^{jk}$.

The Greek letters $\alpha, \beta, \gamma, \delta$ vary from 1 to 5 throughout the paper. Summation convention is applied for both the Greek and Latin indices.In the Miron's frame an arbitrary tensor can be expressed by scalar components along the unit vectors, l^i, m^i, n^i, p^i and q^i For instance, let $T = T^i_j$ be a tensor field of (1, 1) type, then the scalar components $T_{\alpha\beta}$ of T^i_j are defined by

$$T^i_j = T^i_j e_{(\alpha)i}\, e^j_{(\beta)}.$$

and the components T^i_j of the tensor T are expressed as

$$T^i_j = T_{\alpha\beta}\, e^i_{(\alpha)}\, e^i_{(\beta)j}\ .$$

From the equations $g_{ij} e^i_{(\alpha)} e^i_{(\beta)} = \delta_{\alpha\beta}$, we have

(2.1) $$g_{ij} = l_i l_j + m_i m_j + n_i n_j + p_i p_j + q_i q_j$$

Next, the $C-$tensor $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ satisfies $C_{ijk} l^i = 0$ and is symmetric in i, j, k therefore if $C_{\alpha\beta\gamma}$ are scalar components of LC_{ijk}, that is if

(2.2) $$LC_{ijk} = C_{\alpha\beta\gamma}\, e_{(\alpha)i}\, e_{(\beta)j}\, e_{(\gamma)k},$$

then, we have

(2.3) $$\begin{aligned} LC_{ijk} = {} & C_{222} m_i m_j m_k + C_{223}\Pi_{(ijk)}\left(m_i m_j m_k\right) + C_{233}\Pi_{(ijk)}\left(m_i n_j n_k\right) \\ & + C_{333}\left(n_i n_j n_k\right) + C_{224}\Pi_{(ijk)}\left(m_i m_j p_k\right) + C_{444}\left(p_i p_j p_k\right) \\ & + C_{244}\Pi_{(ijk)}\left(m_i p_j p_k\right) + C_{255}\Pi_{(ijk)}\left(m_i m_j q_k\right) + C_{255}\Pi_{(ijk)}\left(m_i q_j q_k\right) \\ & + C_{555}\left(q_i q_j q_k\right) + C_{334}\Pi_{(ijk)}\left(n_i n_j p_k\right) + C_{344}\Pi_{(ijk)}\left(n_i p_j p_k\right) \\ & + C_{335}\Pi_{(ijk)}\left(n_i n_j q_k\right) + C_{355}\Pi_{(ijk)}\left(n_i q_j q_k\right) + C_{445}\Pi_{(ijk)}\left(p_i p_j q_k\right) \\ & + C_{455}\Pi_{(ijk)}\left(p_i q_j q_k\right) + C_{234}\Pi_{(ijk)}\left\{m_i\left(n_j p_k + n_k p_j\right)\right\} \\ & + C_{235}\Pi_{(ijk)}\left\{m_i\left(n_j q_k + n_k q_j\right)\right\} + C_{245}\Pi_{(ijk)}\left\{m_i\left(p_j q_k + p_k q_j\right)\right\} \\ & + C_{345}\Pi_{(ijk)}\left\{n_i\left(p_j q_k + p_k q_j\right)\right\}, \end{aligned}$$

where $\prod_{(ijk)}\{.......\}$ denotes the cyclic interchange of i, j, k and summation. For instance,

$$\Pi_{(ijk)}\left\{A_i B_j C_k\right\} = A_i B_j C_k + A_j B_k C_i + A_k B_i C_j.$$

Contracting (2.2) with g^{ij}, we get $LCM_i = C_{\alpha\beta\beta} C_{(\alpha)i.}$ Thus, if we put

(2.4) $$\begin{cases} C_{222} = H,\ C_{233} = I,\ C_{244} = K,\ C_{333} = J, \\ C_{344} = J',\ C_{444} = H',\ C_{334} = I',\ C_{234} = K', \\ C_{255} = M,\ C_{355} = J'',\ C_{455} = M',\ C_{555} = H'', \\ C_{335} = I'',\ C_{445} = K'',\ C_{235} = N,\ C_{245} = N', \\ C_{345} = M'', \end{cases}$$

then, we have

(2.5)
$$\begin{cases} H+I+K+M=LC,\ C_{233}=-\left(J+J^{'}+J^{''}\right) \\ C_{224}=-\left(H^{'}+I^{'}+M^{'}\right),\ C_{225}=-\left(H^{''}+I^{''}+K^{''}\right). \end{cases}$$

The seventeen scalars $H, I, J, K, H', I', J', K', H'', I'', J'', K'', M, M', M'', N, N'$ are called the main scalars of a five-dimensional Finsler space. We shall use Cartan's connection $C\Gamma = \Gamma^i_{jk},\ G^i_j,\ C^i_{jk}$ in the following section of this paper. The $h-$covariant derivative of the frame field $e_{(\alpha)i}$ are given by[8]

(2.6)
$$e_{(\alpha)i|j} = H_{(\alpha)\beta\gamma}\, e_{(\beta)i}\, e_{(\gamma)j},$$

$H_{(\alpha)\beta\gamma}$, γ being fixed, are given by

(2.7)
$$H_{(\alpha)\beta\gamma} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_r & J_r & k_r \\ 0 & -h_r & 0 & h_r^{'} & J_r^{'} \\ 0 & -J_r & -h_r^{'} & 0 & k_r^{'} \\ 0 & -k_r & -J_r^{'} & -k_r^{'} & 0 \end{bmatrix}.$$

In (2.7), we have put

$$H_{2)3\gamma} = -H_{3)2\,\gamma} = h_r, H_{2)4\gamma} = -H_{4)2\,\gamma} = J_r, H_{2)5\gamma} = -H_{5)2\,\gamma} = k_r,$$
$$H_{3)4\gamma} = -H_{4)3\,\gamma} = h'_r, H_{3)5\gamma} = -H_{5)3\,\gamma} = J'_r, H_{4)5\gamma} = -H_{5)4\,\gamma} = k'_r.$$

Thus, in a five-dimensional Finsler space there exist six $h-$connections vector $h_i, j_i, k_i, h'_i, j'_i, k'_i$ whose scalar components with respect to Miron frame are $h_r, J_r, k_r, H'_r, J'_r, k'_r$ that is

(2.8)
$$\begin{cases} h_i = h_r e_{(r)i},\ J_i = J_r e_{(r)i},\ k_i = k_r e_{(r)i}, \\ h_i^{'} = h_r^{'} e_{(r)i},\ J_i^{'} = J_r e_{(r)i},\ k_i^{'} = k_r^{'} e_{(r)i}. \end{cases}$$

A Finsler space F^n is called C^h-symmetric Finsler if

(2.9)
$$C_{ijk|h} = C_{ijh|k}$$

where | denote $h-$covariant derivative with respect to Cartan's connections.

With the help of equations (2.7) and (2.8), the equations (2.6) can be explicitly written as

(2.10) $$\begin{cases} l_{i|j} = 0,\ m_{i|j} = n_i h_j + p_i J_j + q_i k_j, \\ n_i \mid j = -m_i h_j + p_i h_j^{'} + q_i J_j^{'}, \\ p_{i|j} = -m_i J_j - n_i h_j^{'} + q_i k_j^{'}, \\ q_{i|j} = -m_i k_j - n_i J_j^{'} - p_i k_j^{'}. \end{cases}$$

The $h-$scalar derivative of the adopted components $T_{\alpha\beta}$ of the tensor T_j^i of $(1,1)$ type is defined as[9]

(2.11) $$T_{\alpha\beta,\gamma} = (\delta_k T_{\alpha\beta}) e_{(r)}^k + T_{\mu\beta} H_{(\mu)\alpha\gamma} + T_{\alpha\mu} H_{(\mu)\beta\gamma},$$

where $\delta_k = \partial_k - G_k^r \dot{\partial}_r$.

Thus is $T_{\alpha\beta\gamma}$ is adopted components of $T_{j|k}^i$, that is

(2.12) $$T_{j|k}^i = T_{\alpha\beta,\gamma} e_{(\alpha)}^i e_{(\beta)j} e_{(\gamma)k}$$

From (2.2), it follows that

(2.13) $$LC_{hij|k} = C_{\alpha\beta\gamma,\delta} e_{(\alpha)h} e_{(\beta)i} e_{(\gamma)j} e_{(\delta)k}$$

The explicit form of $C_{\alpha\beta\gamma,\delta}$ is easily obtained:

(2.14) $$\begin{cases} (a)\ C_{1\beta\gamma,\delta} = 0, \\ (b)\ C_{222,\delta} = H_{,\delta} + 3(J + J' + J'')h_\delta + 3(H' + I' + M')J_\delta \\ \qquad + 3(H'' + I'' + K'')k_\delta, \\ (c)\ C_{223,\delta} = -(J + J' + J'')_{,\delta} + (H - 2I)h_\delta - 2K'J_\delta - 2Nk_\delta \\ \qquad + (H' + I' + M')h'_\delta + (H'' + I'' + M'')J'_\delta, \\ (d)\ C_{224,\delta} = -(H' + I' + M')_{,\delta} - 2K'h_\delta + (H - 2K)J_\delta - 2N'k_\delta \\ \qquad - (J + J' + J'')h'_\delta + (H'' + I'' + K'')k'_{\delta,} \\ (e)\ C_{225,\delta} = -(H'' + I'' + K'')_{,\delta} - 2Nh_\delta - 2N'J_\delta + (H - 2M)k_\delta \\ \qquad - (J + J' + J'')J'_\delta - (H' + I' + M')k'_{\delta,} \\ (f)\ C_{233,\delta} = I_{,\delta} - (3J + 2J' + 2J'')h_\delta - I'J_\delta - I''k_\delta - 2NJ'_\delta - 2k'h'_\delta, \\ (g)\ C_{234,\delta} = K'_{,\delta} - (2I' + H' + M')h_\delta - (2J' + J + J'')J_\delta - M''k_\delta \\ \qquad - (Kt\ I)h'_\delta - N'J'_\delta - Nk'_{\delta,} \end{cases}$$

$$
\begin{cases}
(h)\ C_{235,\delta} = N,_{\delta} - (2I'' + H'' + K'')h_{\delta} - M''J_{\delta} - (J + J' + 2J'')k_{\delta} \\
\qquad\qquad - N'h'_{\delta} - (M - I)J'_{\delta} + K'k'_{\delta} \\
(i)\ C_{244,\delta} = K,_{\delta} - J'h_{\delta} - (3H' + 2I' + 2M')J_{\delta} + 2K'h'_{\delta} - K''k_{\delta} - 2N'k'_{\delta}, \\
(j)\ C_{245,\delta} = N',_{\delta} - M''h_{\delta} - (H'' + I'' + 2K'')J_{\delta} + Nh'_{\delta} \\
\qquad\qquad - (H' + I' + 2M')k_{\delta} + K'J'_{\delta} + (K - M)k'_{\delta}, \\
(k)\ C_{255,\delta} = M,_{\delta} - J''h_{\delta} - M'J_{\delta} - (3H'' + 2I'' + 2K'')k_{\delta} \\
\qquad\qquad + 2NJ'_{\delta} + 2N'k'_{\delta}, \\
(l)\ C_{333,\delta} = J,_{\delta} + 3(Ih_{\delta} - I'h'_{\delta} - I''J'_{\delta}), \\
(m)\ C_{334,\delta} = I',_{\delta} + 2K'h_{\delta} + IJ_{\delta} + (J - 2J')h'_{\delta} - 2M''J'_{\delta} - I''k'_{\delta} \\
(n)\ C_{335,\delta} = I'',_{\delta} + 2Nh_{\delta} - 2M''h'_{\delta} + (J - 2J'')J'_{\delta} + Ik_{\delta} + I'k'_{\delta}, \\
(o)\ C_{344,\delta} = J',_{\delta} + Kh_{\delta} + 2K'J_{\delta} - (H - 2I')h'_{\delta} - K''J'_{\delta} - 2M''k'_{\delta}, \\
(p)\ C_{345,\delta} = M'',_{\delta} + N'h_{\delta} + NJ_{\delta} + (I'' - K'')h'_{\delta} + K'k_{\delta} + (I' - M')J'_{\delta} \\
\qquad\qquad + (J' - J'')k'_{\delta}, \\
(q)\ C_{355,\delta} = J'',_{\delta} + Mh_{\delta} - M'h'_{\delta} + 2Nk_{\delta} - (H'' - 2I'')J'_{\delta} + 2M''k'_{\delta}, \\
(r)\ C_{444,\delta} = H',_{\delta} + 3(KJ_{\delta} + J'h'_{\delta} - K''k'_{\delta}), \\
(s)\ C_{445,\delta} = K'',_{\delta} + 2N'J_{\delta} + 2M''h'_{\delta} + Kk_{\delta} + J'J'_{\delta} + (H' - 2M')k'_{\delta}, \\
(t)\ C_{455,\delta} = M',_{\delta} + MJ_{\delta} + J''h'_{\delta} + 2N'k_{\delta} + 2M'J'_{\delta} - (H'' - 2K'')k'_{\delta}, \\
(u)\ C_{555,\delta} = H'',_{\delta} + 3(Mk_{\delta} + J''J'_{\delta} + M'k'_{\delta}).
\end{cases}
$$

where $H,_{\delta}$, for instance is the $h-$scalar derivative of the single scalar H, namely $H_{,\delta} = (\delta_i H)e^i_{(\delta)}$.

Making use of equation (2.9), equation (2.13) yields

$$(2.15) \qquad C_{\alpha\beta\gamma,\delta} - C_{\alpha\beta\delta,\gamma} = 0.$$

This equation is explicitly written as:

$$
(2.16) \qquad
\begin{cases}
(a)\ -(J + J' + J'')_{,2} + (H - 2I)h_2 - 2K'J_2 - 2Nk_2 + \\
\qquad (H' + I' + M')H'_2 + (H'' + I'' + K'')J'_2 \\
\qquad = H,_3 + 3(J + J' + J'')h_3 + 3(H' + I' + M')J_3 \\
\qquad\quad + 3(H'' + I'' + K'')k_3,
\end{cases}
$$

$$\begin{cases}
(b)\ I_{,2}-(3J+2J'+2J'')h_2-I'J_2-I''k_2-2K'h'_2-2NJ_2' \\
\quad =-(J+J'+J'')_{,3}+(H-2I)h_3-2K'J_3-2Nk_3 \\
\quad (H'+I'+M')h'_3+(H''+I''+K'')J'_3, \\
(c)\ K'_{,2}-(2I'+H'+M')h_2-(2J'+J+J'')J_2-M''k_2-(K-I)h'_2 \\
\quad -N'J'_2-Nk'_2=-(J+J'+J'')_{,4}+(H-2I)h_4-2K'J_4 \\
\quad -2Nk_4+(H'+I'+M')h'_4+(H''+I''+K'')J'_4 \\
\quad =-(H'+I'+M')_{,3}-2k'h_3+(H-2K)J_3-2N'k_3 \\
\quad -(J+J'+J'')h'_3+(H''+I''+K'')k'_3, \\
(d)\ N_{,2}-(2I''+H''+K'')h_2-M''J_2-(J+J'+2J'')k_2 \\
\quad -N'h'_2-(M-I)J'_2+K'k'_2 \\
\quad =-(J+J'+J'')_{,5}+(H-2I)h_5-2K'J_5-2Nk_5 \\
\quad +(H'+I'+M')h'_5+(H''+I''+K'')J'_5 \\
\quad =-(H''+I''+K'')_{,3}-2Nh_3-2N'J_3+(H-2M)k_3 \\
\quad -(J+J'+J'')J'_3-(H'+J'+M')k'_3, \\
(e)\ J_{,2}+3(Ih_2-I'h'_2-I''J'_2)=I_{,3}-(3J+2J'+2J'')h_3 \\
\qquad -I'J_3-I''k_3-2k'h'_3-2NJ'_3 \\
(f)\ I'_{,2}+2k'h_2+IJ_2+(J-2J')h'2-2M''J'_2-I''k'_2 \\
\quad =k'_{,3}-(2I'+H'+M')h_3-(2J'+J+J'')J_3-M''k_3 \\
\quad -(K-I)h'_3-N'J'_3-Nk'_3=I_{,4}-(3J+2J'+2J'')h_4 \\
\quad -I'J_4-I''k_4-2K'h_4-2NJ'_4, \\
(g)\ I''_{,2}+2Nh_2-2M''h'_2+(J-2J'')J'_2+Ik_2+I'k'_2 \\
\quad =N_{,3}-(2I''+H''+K'')h_3-M''J_3-(J+J'+2J'')k_3 \\
\quad -N'h'_3-(M-I)J'_3+K'k'_3 \\
\quad =I_{,5}-(3J+2J'+2J'')h_5-I'J_s \\
\quad -I''k_5-2K'h'_5-2NJ'_5,
\end{cases}$$

$$\begin{cases}
(h) & J'_{,2}+Kh_2+2k'J_2-\left(H'-2I'\right)H'_2-K''J'_2-2M''k'_2 \\
& =K_{,3}+J'h_3-\left(3H'+2I'+2M'\right)J_3+2K'h'_3-K''k_3-2N'k'_3 \\
& =K'_{,4}-\left(2I'+H'+M'\right)h_4-(2J'+J+J'')J_4 \\
& -M''k_4-(K-I)h'_4-N'J'_4-Nk'_4\ , \\
(i) & J''_{,2}+Mh_2-M''h'_2+2Nk_2-\left(H''-2I''\right)J'_2+2M''k'_2 \\
& =M_{,3}-J''h_3-M'J_3-\left(3H''+2I''+2K''\right)k_3+2NJ'_3+2N'k'_3 \\
& =N,_5-\left(2I''+H''+K''\right)h_5-M''J_5-\left(J+J'+2J''\right)k_5 \\
& -N'h'_5-\left(M-I\right)J'_5+k'k'_5\ , \\
(j) & M''_{,2}+N'h_2+NJ_2+\left(I''-K''\right)h'+k'k_2+\left(I'-M'\right)J'_2+\left(J'-J''\right)k'_2 \\
& =N'_{,3}-M''h_3-\left(H''+I''+2K''\right)J_3+Nh'_3-\left(H'+I'+2M'\right)k_3 \\
& +K'J'_3+\left(K-M\right)k'_3 \\
& =N_{,4}-\left(2I''+H''+K''\right)h_4-M''J_4-\left(J+J'+2J''\right)k_4-N'h'_4 \\
& -\left(M-I\right)J'_4+k'k'_4 \\
& =K',_5-\left(2I'+H'+M'\right)h_5-\left(2J'+J+J''\right)J_5-M''k_5-\left(K-I\right)h'_5 \\
& -N'J'_5-Nk'_5\ , \\
(k) & -\left(H'+I'+M'\right)_{,2}-2K'h_2+\left(H-2K\right)J_2-2N'k_2-\left(J+J'+J''\right)h'_2 \\
& +\left(H''+I''+K''\right)K'_2 \\
& =H_{,4}+3\left(J+J'+J''\right)h_4+3\left(H'+I'+M'\right)J_4+3\left(H''+I''+K''\right)k_4\ , \\
(l) & K_{,2}+J'h_2-\left(3H'+2I'+2M'\right)J_2+2K'h'_2-K''k_2-2N'k'_2 \\
& =-\left(H'+I'+M'\right)_{,4}-2K'h_4+\left(H-2k\right)J_4-2N'k_4-\left(J+J'+J''\right)h'_4 \\
& +\left(H''+I''+K''\right)k'_4\ , \\
(m) & H'_{,2}+3\left(KJ_2+J'h'_2+K''\right)k'_2=K_{,4}-J'h_4-\left(3H'+2I'+2M'\right)J_4 \\
& \qquad +2K'h_4-K''k_4-2N'k'_4\ , \\
(n) & K''_{,2}+2N'J_2+2M''h'_2+Kk_2+J'J'_2+\left(H'-2M'\right)k'_2 \\
& =N'_{,4}-M''h_4-\left(H''+I''+2K''\right)J_4+Nh'_4-\left(H'+I'+2M'\right)k_4 \\
& +K'J'_4+\left(K-M\right)k'_4=K_{,5}-J'h_5-\left(3H'+2I'+2M'\right)J_5 \\
& +2K'h'_5-K''k_5-2N'k'_5 \\
(o) & M'_{,2}+MJ_2+J''u'_2+2N'k_2+2M''J'_2-\left(H''-2K''\right)k'_2 \\
& =M_{,4}-J''h_4-M'J_4-\left(3H''+2I''+2K''\right)k_4+2NJ'_4 \\
& +2N'k'_4=N'_{,5}-M''h_5-\left(H''+I''+2K''\right)J_5+Nh'_5 \\
& -\left(H'+I'+2M'\right)k_5+K'J'_5+\left(K-M\right)k'_5 \\
(p) & I'_{,3}+2K'h_3+IJ_3+\left(J-2J'\right)h'_3-2M''J'_3-I''k'_3 \\
& =J_{,4}+3\left(Ih_4-I'h'_4-I''J'_4\right),
\end{cases}$$

$$\left\{\begin{array}{l}
(q)\quad J'_{,3}+Kh_3+2K'J_3-\left(H'-2I'\right)h'_3-K''J'_3-2M''k'_3 \\
\qquad =I'_{,4}+2K'h_4+IJ_4+\left(J-2J'\right)h'_4-2M''J'_4-I''k'_4\,, \\
(r)\quad M''_{,3}+N'h_3+NJ_3+\left(I''+K''\right)h'_3+K'k_3+\left(I'-M'\right)J'_3+(J'-J'')k'_3 \\
\qquad =I''_{,4}+2Nh_4-2M''h'_4+\left(J-2J''\right)J'_4+Ik_4+I'k'_4 \\
\qquad =I'_{,5}+2K'h_5+IJ_5+(J-2J')h'_5-2M''J'_5-I''k'_5\,, \\
(s)\quad H'_{,3}+3(KJ_3+J'h_3-K''J'_3-2M''k'_3)=J'_{,4}+Kh_4+2K'J_4-\left(H'-2I'\right)h'_4 \\
\qquad\qquad -K''J'_4-2M''k'_4, \\
(t)\quad T''_{,3}+2N'J_3+2M''h'_3+Kk_3+J'J'_3+\left(H'-2M'\right)k'_3 \\
\qquad =M''_{,4}+N'h_4+NJ_4+\left(I''-K''\right)h_4+K'k_4+(I'-M')J'_4+\left(J'-J''\right)k' \\
\qquad =J'_{,5}+Kh_5+2K'J_5-(H'-2I')h'_5-K''J'_5-2M''k'_5\,, \\
(u)\quad M''_{,3}+MJ_3+J''h'_3+2N'k_3+2M''J'_3-\left(H''-2K''\right)J'_3+(J'-J'')k'_3 \\
\qquad =J''_{,4}+Mh_4-M'h'_4+2Nk_4-\left(H''-2I''\right)J'_4+2M''k'_4 \\
\qquad =M''_{,5}+N'h_5+NJ_5+(I''-K'')h'_5+K'h_5+\left(I'-M'\right)J'_5+\left(J'-J''\right)k'_5\,, \\
(v)\quad -\left(H''+I''+K''\right)_{,2}-2Nh_2-2N'J_2+(H-2M)k_2-\left(J+J'+J''\right)J'_2 \\
\qquad -\left(H'+I'+M'\right)k'_2=H_{,5}+3\left(J+J'+J''\right)h_5+3\left(H'+I'+M'\right)J_5 \\
\qquad\qquad +3\left(H''+I''+K''\right)k_5, \\
(w)\quad N'_{,2}-M''h_2-\left(H''+I''+2K''\right)J_2+Nh'_2-(H'+I'+2M')k_2+K'J'_2 \\
\qquad +\left(K-M\right)k'_2=-(H'+I'+2M')J''_{,4}-2K'h_5-M'h'_4+\left(H-2K\right)J_5 \\
\qquad\qquad -2N'k_5-\left(J+J'+J''\right)h'_5+\left(H''+I''+2K''\right)k'_5 \\
\qquad =-\left(H''+I''+K''\right)_{,4}-2Nh_4-2N'J_4+(H-2M)k_4 \\
\qquad -\left(J+J'+J''\right)J'_4K'h_5-(H'+I'+M')k'_4\,, \\
(x)\quad M_{,2}-J''h_2-\left(3H''+2I''+2K''\right)k_2+2NJ'_2+2N'k'_2 \\
\qquad =-\left(H''+I''+K''\right)_{,5}-2Nh_5-2N'J_5+\left(H-2M\right)k_5 \\
\qquad -\left(J+J'+J''\right)J'_5-\left(H'+I'+M'\right)k'_5\,, \\
(y)\quad H''_{,2}+3\left(MK_2+J''J'_2+M'k'_2\right) \\
\qquad =M_{,5}-J''h_5-M'J_5-\left(3H''+2I''+2K''\right)k_5+2NJ'_5+2N'k'_5\,, \\
(z)\quad I''_{,3}+2Nh_3-2M''h'_3+\left(J-2J''\right)J'_3+Ik_3+I'k'_3 \\
\qquad =J_{,5}+3\left(Ih_5-I'h'_5-I''J'_5\right),
\end{array}\right.$$

$$\begin{cases} (A) & J''_{,3} + Mh_3 - M'h'_3 + 2Nk_3 - \left(H'' - 2I''\right)J'_3 + 2M''k'_3 \\ & \quad = I''_{,5} + 2Nh_5 - 2M''h'_5 + \left(J - 2J''\right)J'_5 + Ik_5 + I'k'_5 \ , \\ (B) & H''_{,3} + 3\left(Mk_3 + J''J'_3 + M'k'_3\right) \\ & \quad = J''_{,5} + Mh_5 - M'h'_5 + 2Nk_5 - \left(H'' - 2I''\right)J'_5 + 2M''k'_5 \ , \\ (C) & K''_{,4} + 2N'J_4 + 2M''h'_4 + J'J'_4 - \left(C' - 2M'\right)k'_4 \\ & \quad = H'_{,5} + 3\left(KJ_5 + J'h'_5 - K''k'_5\right), \\ (D) & M'_{,4} + MJ_4 + J''h'_4 + 2N'k_4 + 2M''J'_4 - \left(H'' - 2K''\right)k'_4 \\ & \quad = K''_{,5} + 2N'J_5 + 2M''h'_5 + Kk_5 + J'J'_5 + \left(H' - 2M'\right)k'_5 \ , \\ (E) & H''_{,4} + 3\left(Mk_4 + J''J'_4 + M'k'_4\right) \\ & \quad = M'_{,5} + MJ_5 + J''h'_5 + 2N'k_5 + 2M''J'_5 - \left(H'' - 2K''\right)k'_5 \ . \end{cases}$$

Since $C_{ijh}y^h = 0$ and $y^h_{|k} = 0$. Hence from (2.9) it follows that $C_{ijk|h}y^h = 0$, that is $P_{ijk} = 0$. Therefore, we have the following theorem:

Theorem 2.1: *A five-dimensional $C^h -$symmetric Finsler space with unified main scalar is a Landsberg space.*

The converse of theorem (2.1) is not necessarily true, so the $C^h -$ symmetric Finsler space is more general than the Landsberg space.

3. The Constant Unified Main Scalar

In a five-dimensional Finsler space, $H + I + K + M = LC$ is called the unified main scalar. Now, we consider five-dimensional $C^h -$symmetric Finsler space with non-zero constant unified main scalar. Therefore, we have

(3.1) $\quad \left(H + I + K + M\right),\alpha = \left(LC\right),\alpha = 0$ for $\alpha = 1, 2, 3, 4, 5$.

Adding equation (2.16 (a), (2.16) (e), first part of equation (2.16) (h) and first part of equation (2.16) (i) and applying equation (3.1), we get $h_2 = 0.$ Similarly, adding equations (2.16) (k), (2.16) (m), first part of equation (2.16) (f) and first part of equation (2.16) (o) and applying equation (3.1), we get $J_2 = 0.$ Again adding equations (2.16)(b), (2.16)(s), first part of equation (2.16) (u) and last part of equation (2.16) (c) and applying equation (3.1), we get $J_3 = h_4.$ Hence we have the following:

Theorem 3.1: *In a five-dimensional C^h – symmetric Finsler space with non-zero constant unified main scalar, the scalar components of h – connection vectors h_i and J_i are given by*

$$h_i = h_1 l_i + h_3 n_i + h_4 p_i + h_5 q_i,$$
$$J_i = J_1 l_i + J_3 n_i + J_4 p_i + J_5 q_i,$$

where $J_3 = h_4$.

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