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Four-dimensional Finsler spaces with *T*-tensor of some special forms

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Abstract: The *T*-tensor played an important role in the Finsler geometry. In this paper, we discuss a four-dimensional Finsler space whose *T*-tensor is of special forms.

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1. Orthonormal frame and connection vectors

Let M^4 be a four-dimensional smooth manifold and $F^4=(M^4,L)$ be a four-dimensional Finsler space equipped with a metric function $L(x,y)$ on M^4. The normalized supporting element, the metric tensor, the angular metric tensor and Cartan tensor are defined by

$$l_i=\dot{\partial}_i L, \quad g_{ij}=\frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \quad h_{ij}=L\dot{\partial}_i\dot{\partial}_j L \quad \text{and} \quad C_{ijk}=\frac{1}{2}\dot{\partial}_k g_{ij}$$

respectively.

The torsion vector C^i is defined by $C^i=C^i_{jk}g^{jk}$. Throughout this paper, we use the symbols $\dot{\partial}_i$ and ∂_i for $\partial/\partial y^i$ and $\partial/\partial x^i$ respectively. The Cartan connection in the Finsler space is given as $C\Gamma=(F^i_{jk},G^i_j,C^i_{jk})$. The *h*- and *v*-covariant derivatives of a covariant vector $X_i(x,y)$ with respect to the Cartan connection are given by\

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(1.1) $$X_{i|j} = \partial_j X_i - (\dot{\partial}_h X_i) G^h_j - F^r_{ij} X_r,$$

and

(1.2) $$X_i|_j = \dot{\partial}_j X_i - C^r_{ij} X_r.$$

In 1972, H. Kawaguchi[1] and M. Matsumoto[2] independently found an important tensor

(1.3) $$T_{hijk} = L C_{hij}|_k + C_{hij} l_k + C_{hik} l_j + C_{hkj} l_i + C_{kij} l_h.$$

This is called the *T*-tensor. It is completely symmetric in its indices. The vanishing of *T*-tensor is called *T*-condition.

U.P. Singh et al.[3-4] studied three-dimensional Finsler spaces with *T*-tensor of the following forms:

(A) $$T_{hijk} = \rho (h_{hi} h_{jk} + h_{hj} h_{ik} + h_{hk} h_{ij}),$$

(B) $$T_{hijk} = h_{hi} P_{jk} + h_{hj} P_{ik} + h_{hk} P_{ij} + h_{ij} P_{hk} + h_{ik} P_{hj} + h_{jk} P_{hi},$$

(C) $$T_{hijk} = \rho C_h C_i C_j C_k + a_h C_i C_j C_k + a_i C_h C_j C_k + a_j C_h C_i C_k + a_k C_h C_i C_j,$$

where P_{ij} are the components of a tensor field, a_h are the components of a covariant vector field and ρ is a scalar. The present authors[6-10] studied the theory of four-dimensional Finsler space. In this paper, we discuss four-dimensional Finsler spaces with *T*-tensor of such forms.

2. Four-dimensional Finsler space

The Miron frame for a four-dimensional Finsler space is constructed by the unit vectors $(e^i_{1)}, e^i_{2)}, e^i_{3)}, e^i_{4)})$. The first vector $e^i_{1)}$ is the normalized supporting element l^i and the second $e^i_{2)}$ is the normalized torsion vector $m^i = C^i / \mathbb{C}$, the third $e^i_{3)} = n^i$ and the fourth $e^i_{4)} = p^i$ are constructed by $g_{ij} e^i_{\alpha)} e^j_{\beta)} = \delta_{\alpha\beta}$. We suppose that the length $\mathbb{C}$ of the vector C^i does not vanish, i.e. the space is non-Riemannian. With respect to this frame, the scalar components of an arbitrary tensor T^i_j are defined by

(2.1) $$T_{\alpha\beta} = T^i_j e_{\alpha)i} e^j_{\beta)},$$

from which, we get

(2.2) $$T^i_j = T_{\alpha\beta} e^i_{\alpha)} e_{\beta)j},$$

where summation convention is also applied to Greek indices. The scalar components of the metric tensor g_{ij} are $\delta_{\alpha\beta}$.

Let $H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}/L$ be scalar components of the *h*- and *v*-covariant derivatives $e^i_{\alpha)|j}$ and $e^i_{\alpha)}|_j$ respectively of the vectors $e^i_{\alpha)}$, then

(2.3) $$e^i_{\alpha)|j} = H_{\alpha)\beta\gamma} e^i_{\beta)} e_{\gamma)j},$$

and

(2.4) $$Le^i_{\alpha)}|_j = V_{\alpha)\beta\gamma} e^i_{\beta)} e_{\gamma)j}.$$

$H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}$ are called *h*- and *v*-connection scalars respectively and are positively homogeneous of degree zero in *y*. Orthogonality of the Miron frame yields[5] $H_{\alpha)\beta\gamma} = -H_{\beta)\alpha\gamma}$ and $V_{\alpha)\beta\gamma} = -V_{\beta)\alpha\gamma}$. Also we have $H_{1)\beta\gamma} = 0$ and $V_{1)\beta\gamma} = \delta_{\beta\gamma} - \delta_{1\beta}\delta_{1\gamma}$.

Now we define Finsler vector fields:

$$h_i = H_{2)3\gamma} e_{\gamma)i}, \quad j_i = H_{4)2\gamma} e_{\gamma)i}, \quad k_i = H_{3)4\gamma} e_{\gamma)i},$$

and

$$u_i = V_{2)3\gamma} e_{\gamma)i}, \quad v_i = V_{4)2\gamma} e_{\gamma)i}, \quad w_i = V_{3)4\gamma} e_{\gamma)i}.$$

The vector fields h_i, j_i, k_i are called *h*-connection vectors and the vector fields u_i, v_i, w_i are called *v*-connection vectors[6-10]. The scalars $H_{2)3\gamma}$, $H_{4)2\gamma}$, $H_{3)4\gamma}$ and $V_{2)3\gamma}$, $V_{4)2\gamma}$, $V_{3)4\gamma}$ are considered as the scalar components $h_\gamma, j_\gamma, k_\gamma$ and $u_\gamma, v_\gamma, w_\gamma$ of the *h*- and *v*-connection vectors respectively with respect to the orthonormal frame.

From (2.4), we get

(2.5)
$$\begin{cases} \text{a)} & L\, e^i_{1)}|_j = L\, l^i|_j = m^i m_j + n^i n_j + p^i p_j = h^i_j, \\ \text{b)} & L\, e^i_{2)}|_j = L m^i|_j = -l^i m_j + n^i u_j - p^i v_j, \\ \text{c)} & L e^i_{3)}|_j = L n^i|_j = -l^i n_j - m^i u_j + p^i w_j, \\ \text{d)} & L e^i_{4)}|_j = L p^i|_j = -l^i p_j + m^i v_j - n^i w_j. \end{cases}$$

Because of the homogeneity of $e^i_{\alpha)}$, (2.5) gives

$$L m^i|_j\, l^j = 0 = n^i u_j l^j - p^i v_j l^j,$$

$$L n^i |_j \, l^j = 0 = -m^i u_j l^j + p^i w_j l^j .$$

These imply $u_1 = u_j l^j = 0,\ v_1 = v_j l^j = 0,\ w_1 = w_j l^j = 0$. Thus, we have:

Proposition 2.1- *The first scalar components* u_1, v_1 *and* w_1 *of the v-connection vectors* u_i, v_i, w_i *vanish identically.*

Let $C_{\alpha\beta\gamma}$ be scalar components of LC_{ijk} with respect to the Miron frame, i.e.

(2.6) $$LC_{ijk} = C_{\alpha\beta\gamma} e_{\alpha)i} e_{\beta)j} e_{\gamma)k} .$$

The main scalars of a four-dimensional Finsler space are given by[6-8]

$$C_{222} = A,\ C_{233} = B,\ C_{244} = C,\ C_{322} = D,$$

$$C_{333} = E,\ C_{422} = F,\ C_{433} = G,\ C_{234} = H .$$

We also have $C_{344} = -(D+E)$, $C_{444} = -(F+G)$ and

(2.7) $$A + B + C = L\mathfrak{C} .$$

The scalar components $T_{\alpha\beta;\gamma}$ of $L\, T^i_j |_k$ are written in the form[5]

(2.8) $$T_{\alpha\beta;\gamma} = L(\dot{\partial}_k T_{\alpha\beta}) e^k_{\gamma)} + T_{\mu\beta} V_{\mu)\alpha\gamma} + T_{\alpha\mu} V_{\mu)\beta\gamma} .$$

The explicit form of $C_{\alpha\beta\gamma;\delta}$ is obtained as follows

(2.9) $$\begin{cases} C_{222;\delta} = A_{;\delta} - 3D u_\delta + 3F v_\delta ; \\ C_{233;\delta} = B_{;\delta} + (2D-E) u_\delta + G v_\delta - 2H w_\delta , \\ C_{244;\delta} = C_{;\delta} + (D+E) u_\delta - (3F+G) v_\delta + 2H w_\delta , \\ C_{322;\delta} = D_{;\delta} + (A-2B) u_\delta + 2H v_\delta - F w_\delta , \\ C_{333;\delta} = E_{;\delta} + 3B u_\delta - 3G w_\delta , \\ C_{422;\delta} = F_{;\delta} - 2H u_\delta - (A-2C) v_\delta + D w_\delta , \\ C_{433;\delta} = G_{;\delta} + 2H u_\delta - B v_\delta + (2D+3E) w_\delta , \\ C_{234;\delta} = H_{;\delta} + (F-G) u_\delta - (2D+E) v_\delta + (B-C) w_\delta , \\ C_{344;\delta} = -D_{;\delta} - E_{;\delta} + C u_\delta - 2H v_\delta + (F+3G) w_\delta , \\ C_{444;\delta} = -F_{;\delta} - G_{;\delta} - 3C v_\delta - (3D+3E) w_\delta , \\ C_{1\beta\gamma;\delta} = -C_{\beta\gamma\delta} . \end{cases}$$

where $A_{;\delta}=L(\dot{\partial}_k A)e^k_{\delta)}$. From (2.7) and (2.9), we get

$$\text{(2.10)}\qquad \begin{cases} C_{222;\delta}+C_{233;\delta}+C_{244;\delta}=A_{;\delta}+B_{;\delta}+C_{;\delta}=(L\mathbb{C})_{;\delta}\,, \\ C_{322;\delta}+C_{333;\delta}+C_{344;\delta}=(A+B+C)u_\delta=L\mathbb{C}u_\delta\,. \\ C_{422;\delta}+C_{433;\delta}+C_{444;\delta}=-(A+B+C)v_\delta=-L\mathbb{C}v_\delta\,, \end{cases}$$

From (2.6), it follows that

$$L^2 C_{hij}|_k + LC_{hij}l_k = C_{\alpha\beta\gamma;\delta}e_{\alpha)h}e_{\beta)i}e_{\gamma)j}e_{\delta)k}\,,$$

which implies

$$\text{(2.11)}\qquad L^2 C_{hij}|_k=(C_{\alpha\beta\gamma;\delta}-C_{\alpha\beta\gamma}\delta_{1\delta})e_{\alpha)h}e_{\beta)i}e_{\gamma)j}e_{\delta)k}\,.$$

From (1.3) and (2.11), we get

$$\text{(2.12)}\qquad LT_{hijk}=(C_{\alpha\beta\gamma;\delta}+C_{\beta\gamma\delta}\delta_{1\alpha}+C_{\alpha\gamma\delta}\delta_{1\beta}+C_{\alpha\beta\delta}\delta_{1\gamma})e_{\alpha)h}e_{\beta)i}e_{\gamma)j}e_{\delta)k}\,.$$

Since the tensor $C_{hij}|_k$ is completely symmetric in its indices, from (2.11) we get

$$\text{(2.13)}\qquad C_{\alpha\beta\gamma;\delta}-C_{\alpha\beta\delta;\gamma}=C_{\alpha\beta\gamma}\,\delta_{1\delta}-C_{\alpha\beta\delta}\,\delta_{1\gamma}\,.$$

In view of (2.13), equation (2.10) gives

$$\text{(2.14)}\qquad \begin{aligned} L\mathbb{C}\,u_2 &= C_{322;2}+C_{333;2}+C_{344;2}=C_{222;3}+C_{233;3}+C_{244;3}=(L\mathbb{C})_{;3}\,, \\ -L\mathbb{C}\,v_2 &= C_{422;2}+C_{433;2}+C_{444;2}=C_{222;4}+C_{233;4}+C_{244;4}=(L\mathbb{C})_{;4}\,, \\ L\mathbb{C}\,u_4 &= C_{322;4}+C_{333;4}+C_{344;4}=C_{422;3}+C_{433;3}+C_{444;3}=-L\mathbb{C}v_3\,, \end{aligned}$$

Since $L_{;3}=L(\dot{\partial}_i L)e^i_{3)}=Ll_in^i=0$ and $L_{;4}=L(\dot{\partial}_i L)e^i_{4)}=Ll_ip^i=0$, we have:

Proposition 2.2- *The scalar components u_2 and v_2 of the v-connection vectors u_i and v_i of a four-dimensional Finsler space are given by*

$$u_2=\mathbb{C}^{-1}\,\mathbb{C}_{;3}\,,\qquad v_2=-\mathbb{C}^{-1}\,\mathbb{C}_{;4}\,,$$

and the scalar components u_4 and v_3 are related by $u_4=-v_3$.

3. *T*-tensor of form (A)

A Finsler space is C-reducible if and only if the *T*-tensor is of the form (A) for $\rho\neq 0$ [11-12]. Let F^4 be a four-dimensional Finsler space with *T*-tensor of the form (A). The scalar components of the angular metric tensor h_{ij} are given by

$$h_{ij} = (\delta_{\alpha\beta} - \delta_{1\alpha}\delta_{1\beta})\, e_{\alpha)i}\, e_{\beta)j}\,,$$

therefore in view of (2.12) and (A), we have

$$\begin{aligned} C_{\alpha\beta\gamma;\delta} + C_{\beta\gamma\delta}\,\delta_{1\alpha} + C_{\alpha\gamma\delta}\,\delta_{1\beta} + C_{\alpha\beta\delta}\,\delta_{1\gamma} = \rho L\{ & (\delta_{\alpha\beta} - \delta_{1\alpha}\delta_{1\beta})(\delta_{\gamma\delta} - \delta_{1\gamma}\delta_{1\delta}) \\ & + (\delta_{\alpha\gamma} - \delta_{1\alpha}\delta_{1\gamma})(\delta_{\beta\delta} - \delta_{1\beta}\delta_{1\delta}) \\ & + (\delta_{\alpha\delta} - \delta_{1\alpha}\delta_{1\delta})(\delta_{\beta\gamma} - \delta_{1\beta}\delta_{1\gamma\beta})\}, \end{aligned}$$

which gives

$$\text{(3.1)}\quad \left\{ \begin{array}{lll} C_{222;\delta} = 3\rho L\delta_{2\delta}\,, & C_{233;\delta} = \rho L\delta_{2\delta}\,, & C_{244;\delta} = \rho L\delta_{2\delta}\,, \\ C_{322;\delta} = \rho L\delta_{3\delta}\,, & C_{333;\delta} = 3\rho L\delta_{3\delta}\,, & C_{344;\delta} = \rho L\delta_{3\delta}\,, \\ C_{422;\delta} = \rho L\delta_{4\delta}\,, & C_{433;\delta} = \rho L\delta_{4\delta}\,, & C_{444;\delta} = 3\rho L\delta_{4\delta}\,. \end{array} \right.$$

Putting (3.1) into (2.10), we get

$$(L\mathbb{C})_{;\delta} = 5\rho L\delta_{2\delta}\,,$$

$$L\mathbb{C}\, u_\delta = 5\rho L\delta_{3\delta}\,,$$

$$-L\mathbb{C}\, v_\delta = 5\rho L\delta_{4\delta}\,.$$

Also from the first equation of (3.1), we get

$$C_{222;\delta} = A_{;\delta} - 3D u_\delta + 3F v_\delta = 3\rho L\delta_{2\delta}\,.$$

Thus, we have:

Theorem 3.1- *If the T-tensor of a four-dimensional Finsler space is of the form* (*A*) *then* ρ *is given by*

$$\rho = \frac{A_{;2}}{3L} = \frac{1}{5}\mathbb{C}_{;2} = \frac{1}{5}\mathbb{C}\, u_3 = -\frac{1}{5}\mathbb{C}\, v_4\,.$$

Theorem 3.2- *The scalar components of v-connection vectors* u_i *and* v_i *of a four-dimensional Finsler space with T-tensor of the form* (*A*), *are given by*

$u_1 = 0, \quad u_2 = 0, \quad u_3 = \mathfrak{C}^{-1}\,\mathfrak{C}_{;2}\,, \quad u_4 = 0,$

$v_1 = 0, \quad v_2 = 0, \quad v_3 = 0, \quad v_4 = -\,\mathfrak{C}^{-1}\,\mathfrak{C}_{;2}\,.$

4. *T*-tensor of form (B)

Ikeda[13] showed that for an *n*-dimensional Finsler space with *T*-tensor of the form

(B) $$T_{hijk} = h_{hi}\,P_{jk} + h_{hj}\,P_{ik} + h_{hk}\,P_{ij} + h_{ij}\,P_{hk} + h_{ik}\,P_{hj} + h_{jk}\,P_{hi}\,,$$

we get

$$P_{ij} = \frac{1}{n+3}\left\{T_{ij} - \frac{T}{2(n+1)}h_{ij}\right\},$$

where $T_{ij} = T_{hijk}\,g^{hk}$ and $T = T_{ij}\,g^{ij}$. Therefore (B) becomes

$$T_{hijk} = \frac{1}{n+3}\left(h_{hi}\,T_{jk} + h_{hj}\,T_{ik} + h_{hk}\,T_{ij} + h_{ij}\,T_{hk} + h_{ik}\,T_{hj} + h_{jk}\,T_{hi}\right) - \frac{T}{(n+1)(n+3)}(h_{hi}\,h_{jk} + h_{hj}\,h_{ik} + h_{hk}\,h_{ij}).$$

Thus for a four-dimensional Finsler space, we have

(4.1) $$T_{hijk} = \frac{1}{7}\Big[\left(h_{hi}\,T_{jk} + h_{hj}\,T_{ik} + h_{hk}\,T_{ij} + h_{ij}\,T_{hk} + h_{ik}\,T_{hj} + h_{jk}\,T_{hi}\right) - \frac{T}{5}(h_{hi}\,h_{jk} + h_{hj}\,h_{ik} + h_{hk}\,h_{ij})\Big]$$

Let $T_{\alpha\beta}$ be the scalar components of LT_{hi}, i.e.

$$LT_{hi} = T_{\alpha\beta}\,e_{\alpha)h}\,e_{\beta)i}\,.$$

In view of (2.12) and (4.1), we get

$$\begin{aligned} C_{\alpha\beta\gamma;\delta} + C_{\beta\gamma\delta}\,\delta_{1\alpha} + C_{\alpha\gamma\delta}\,\delta_{1\beta} + C_{\alpha\beta\delta}\,\delta_{1\gamma} = \frac{1}{7}\Big[&\big\{(\delta_{\alpha\beta} - \delta_{1\alpha}\delta_{1\beta})T_{\gamma\delta} + (\delta_{\alpha\gamma} - \delta_{1\alpha}\delta_{1\gamma})T_{\beta\delta} \\ &+ (\delta_{\alpha\delta} - \delta_{1\alpha}\delta_{1\delta})T_{\beta\gamma} + (\delta_{\beta\gamma} - \delta_{1\beta}\delta_{1\gamma})T_{\alpha\delta} + (\delta_{\beta\delta} - \delta_{1\beta}\delta_{1\delta})T_{\alpha\gamma} \\ &+ (\delta_{\gamma\delta} - \delta_{1\gamma}\delta_{1\delta})T_{\alpha\beta}\big\} - \frac{LT}{5}\big\{(\delta_{\alpha\beta} - \delta_{1\alpha}\delta_{1\beta})(\delta_{\gamma\delta} - \delta_{1\gamma}\delta_{1\delta}) \\ &+ (\delta_{\alpha\gamma} - \delta_{1\alpha}\delta_{1\gamma})(\delta_{\beta\delta} - \delta_{1\beta}\delta_{1\delta}) + (\delta_{\alpha\delta} - \delta_{1\alpha}\delta_{1\delta})(\delta_{\beta\gamma} - \delta_{1\beta}\delta_{1\gamma\beta})\big\}\Big], \end{aligned}$$

which gives

$$
(4.2)\qquad \begin{cases}
C_{222;\delta} = \dfrac{1}{7}\left\{3T_{2\delta} + 3T_{22}\delta_{2\delta} - \dfrac{3}{5}LT\delta_{2\delta}\right\} \\
C_{233;\delta} = \dfrac{1}{7}\left\{T_{33}\delta_{2\delta} + T_{2\delta} + 2T_{23}\delta_{3\delta} - \dfrac{1}{5}LT\delta_{2\delta}\right\} \\
C_{244;\delta} = \dfrac{1}{7}\left\{T_{44}\delta_{2\delta} + T_{2\delta} + 2T_{24}\delta_{4\delta} - \dfrac{1}{5}LT\delta_{2\delta}\right\} \\
C_{322;\delta} = \dfrac{1}{7}\left\{T_{22}\delta_{3\delta} + T_{3\delta} + 2T_{23}\delta_{2\delta} - \dfrac{1}{5}LT\delta_{3\delta}\right\} \\
C_{333;\delta} = \dfrac{1}{7}\left\{3T_{3\delta} + 3T_{33}\delta_{3\delta} - \dfrac{3}{5}LT\delta_{3\delta}\right\} \\
C_{344;\delta} = \dfrac{1}{7}\left\{T_{44}\delta_{3\delta} + T_{3\delta} + 2T_{34}\delta_{4\delta} - \dfrac{1}{5}LT\delta_{3\delta}\right\} \\
C_{422;\delta} = \dfrac{1}{7}\left\{T_{22}\delta_{4\delta} + T_{4\delta} + 2T_{24}\delta_{2\delta} - \dfrac{1}{5}LT\delta_{4\delta}\right\} \\
C_{433;\delta} = \dfrac{1}{7}\left\{T_{33}\delta_{4\delta} + T_{4\delta} + 2T_{34}\delta_{3\delta} - \dfrac{1}{5}LT\delta_{4\delta}\right\} \\
C_{444;\delta} = \dfrac{1}{7}\left\{3T_{4\delta} + 3T_{44}\delta_{4\delta} - \dfrac{3}{5}LT\delta_{4\delta}\right\}.
\end{cases}
$$

Putting (4.2) into (2.10), we get

$$
\begin{aligned}
(L\mathbb{C})_{;\delta} &= \frac{1}{7}\{5T_{2\delta} + (3T_{22} + T_{33} + T_{44} - LT)\delta_{2\delta} + 2T_{23}\delta_{3\delta} + 2T_{24}\delta_{4\delta}\}, \\
L\mathbb{C}u_{\delta} &= \frac{1}{7}\{5T_{3\delta} + 2T_{23}\,\delta_{2\delta} + (T_{22} + 3T_{33} + T_{44} - LT)\,\delta_{3\delta} + 2T_{34}\delta_{4\delta}\}, \\
-L\mathbb{C}\ v_{\delta} &= \frac{1}{7}\{5T_{4\delta} + 2T_{24}\,\delta_{2\delta} + 2T_{34}\delta_{3\delta} + (T_{22} + T_{33} + 3T_{44} - LT)\delta_{4\delta}\}.
\end{aligned}
$$

Therefore

$$
(4.3)\qquad \begin{cases}
(L\mathbb{C})_{;2} = \dfrac{1}{7}\{8T_{22} + T_{33} + T_{44} - LT\}, \quad (L\mathbb{C})_{;3} = T_{23}, \quad (L\mathbb{C})_{;4} = T_{24}, \\
L\mathbb{C}u_2 = T_{23}, \quad L\mathbb{C}u_3 = \dfrac{1}{7}\{T_{22} + 8T_{33} + T_{44} - LT\}, \quad L\mathbb{C}u_4 = T_{34}, \\
-L\mathbb{C}v_2 = T_{24}, \quad -L\mathbb{C}v_3 = T_{34}, \quad -L\mathbb{C}v_4 = \dfrac{1}{7}\{T_{22} + T_{33} + 8T_{44} - LT\}.
\end{cases}
$$

From $T = T_{ij}g^{ij}$, we find

$$LT = T_{\alpha\beta}\,\delta_{\alpha\beta} = T_{\alpha\alpha} = T_{22} + T_{33} + T_{44}\,.$$

Thus, in view of (4.3), we have:

Theorem 4.1- *If the T-tensor of a four-dimensional Finsler space is of the form (B), the scalar components of the tensor T_{ij} are given by*

$$T_{1\alpha} = 0\,, \quad T_{22} = (L\mathbb{C})_{;2}\,, \quad T_{33} = L\,\mathbb{C}\,u_3\,, \quad T_{44} = -L\,\mathbb{C}\,v_4\,,$$

$$T_{23} = L\,\mathbb{C}\,u_3 = (L\mathbb{C})_{;3}\,, \quad T_{24} = -L\,\mathbb{C}\,v_2 = (L\mathbb{C})_{;4}\,, \quad T_{34} = L\,\mathbb{C}\,u_4 = -L\,\mathbb{C}\,v_3\,,$$

and $T = \mathbb{C}_{;2} + \mathbb{C}\,u_3 - \mathbb{C}\,v_4$.

5. *T*-tensor of form (C)

U. P. Singh et al.[4] showed that the *T*-tensor of a C-2 like Finsler space is of the form

(C) $$T_{hijk} = \rho\, C_h C_i C_j C_k + a_h C_i C_j C_k + a_i C_h C_j C_k + a_j C_h C_i C_k + a_k C_h C_i C_j\,.$$

Let a_α be the scalar components of $L a_i$, i.e.

$$L a_i = a_\alpha\, e_{\alpha)i}\,.$$

Since $e_{2)i} = C_i / \mathbb{C}$, we get $C_i = \mathbb{C}\;\delta_{2\alpha}\, e_{\alpha)i}$. Therefore in view of (2.12) and (C), we have

$$C_{\alpha\beta\gamma;\delta} + C_{\beta\gamma\delta}\,\delta_{1\alpha} + C_{\alpha\gamma\delta}\delta_{1\beta} + C_{\alpha\beta\delta}\delta_{1\gamma} = \rho\, L\,\mathbb{C}^4\,\delta_{2\alpha}\delta_{2\beta}\delta_{2\gamma}\delta_{2\delta}$$
$$+\mathbb{C}^3\,(a_\alpha\delta_{2\beta}\delta_{2\gamma}\delta_{2\delta} + a_\beta\delta_{2\alpha}\delta_{2\gamma}\delta_{2\delta} + a_\gamma\delta_{2\alpha}\delta_{2\beta}\delta_{2\delta} + a_\delta\delta_{2\alpha}\delta_{2\beta}\delta_{2\gamma})\,,$$

which gives

(5.1) $$\begin{cases} C_{222;\delta} = \mathbb{C}^3\,(\rho\, L\,\mathbb{C} + 3a_2)\,\delta_{2\delta} + \mathbb{C}^3\, a_\delta\,, & C_{233;\delta} = 0\,, \quad C_{244;\delta} = 0\,, \\ C_{322;\delta} = \mathbb{C}^3\, a_3\delta_{2\delta}\,, & C_{333;\delta} = 0\,, \quad C_{344;\delta} = 0\,, \\ C_{422;\delta} = \mathbb{C}^3\, a_4\delta_{2\delta}\,, & C_{433;\delta} = 0\,, \quad C_{444;\delta} = 0\,. \end{cases}$$

Putting (5.1) into (2.10), we get

$$(L\mathfrak{C})_{;\delta} = \mathfrak{C}^3(\rho L\mathfrak{C}+3a_2)\delta_{2\delta} + \mathfrak{C}^3 a_\delta,$$
$$L\mathfrak{C}u_\delta = \mathfrak{C}^3 a_3\delta_{2\delta},$$
$$-L\mathfrak{C}v_\delta = \mathfrak{C}^3 a_4\delta_{2\delta}.$$

Since T_{hijk} is an indicatory tensor, from (C) it follows that $a_1 = a_i\, y^i = 0$. Thus we have:

Theorem 5.1- *If the T-tensor of a four-dimensional Finsler space is of the form (C), the scalar components a_α of the La_i are given by*

$$a_1 = 0, \qquad a_2 = \frac{L}{4}(\mathfrak{C}^{-3}\,\mathfrak{C}_{;2} - \rho\mathfrak{C}),$$
$$a_3 = L\mathfrak{C}^{-2}u_2 = \mathfrak{C}^{-3}(L\mathfrak{C})_{;3}, \qquad a_4 = -L\,\mathfrak{C}^{-2}v_2 = \mathfrak{C}^{-3}(L\mathfrak{C})_{;4}.$$

Theorem 5.2- *In a four-dimensional Finsler space with T-tensor of the form (C), the scalar components of v-connection vectors u_i and v_i are given by*

$$L\mathfrak{C}u_\delta = \mathfrak{C}^3 a_3\delta_{2\delta}, \qquad -L\mathfrak{C}v_\delta = \mathfrak{C}^3 a_4\delta_{2\delta}.$$

Corollary 5.1- *In a four-dimensional Finsler space with T-tensor of the form (C), the v-connection vectors u_i and v_i vanish if the scalar components a_3 and a_4 of La_i vanish.*

6. T-2 like Finsler space

A non-Riemannian Finsler space $F^n (n > 2)$ is called *T*-2 like Finsler space if the *T*-tensor T_{hijk} is written in the form

(6.1) $$T_{hijk} = \rho C_h\, C_i\, C_j\, C_k\,.$$

Equation (6.1) is a particular case of (C) when $a_i = 0$. Thus we have:

Theorem 6.1- *In a T-2 like four-dimensional Finsler space, the v-connection vectors u_i and v_i vanish.*

Theorem 6.2- *In a T-2 like four-dimensional Finsler space, ρ is given by*

$$\rho = \mathfrak{C}^{-4}\ \mathfrak{C}_{;2}\,.$$

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