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A Note on the |N,p,q| Summability of a Factored Fourier Series

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Abstract: In this paper I have proved a theorem on |N,p,q| summability of a factored Fourier series, which generalizes various known results. However the theorem is as follows.

Theorem: If $\Phi(t)$ is a function of bounded variation in $(0,\pi)$ then the factored Fourier series.

$$\sum \frac{(p*q)_n \lambda_n}{n^{\alpha}} A_n(t)$$

is $|N,p,q|$ summable at $t=x$ where the sequences $\{p_n\}$ and $\{q_n\}$ are non-negative non-increasing such that

(i) $\left\{\frac{(n+1)p_n}{(p*q)_n}\right\}$ is of bounded variation

(ii) $\left\{\frac{(n+1)q_n}{(p*q)_n}\right\}$ is of bounded variation

(iii) $\left\{\frac{(p*q)_n}{n^{\alpha}}\right\}$ is bounded

(iv) $\left\{\frac{p_{n+1}}{p_n}\right\}$ is non-decreasing

(v) $\left\{\frac{q_{n+1}}{q_n}\right\}$ is non-decreasing

and $\{\lambda_n\}$ is a convex sequence such that $\sum \frac{\lambda_n}{n} < \infty$.

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1. Definitions and Notations

1. Let $\sum a_n$ be a given infinite series. We denote by $\{s_n\}$, the sequence of its partial sum, i.e. $s_n = \sum_{i=0}^{n} a_i$.

Let $\{p_n\}$ and $\{q_n\}$ be sequences of constants, real or complex, and write

$$(p*q)_n = p_0 q_n + p_1 q_{n-1} + \ldots\ldots\ldots\ldots + p_n q_0$$
$$= \sum_{i=0}^{n} p_{n-i} q_i .$$

We assume throughout that $(p*q)_n \neq 0$, when $n \geq 0$ and $(p*q)_n = 0$ when $n < 0$.

The $n^{th}(N,p,q)$ transform by Borwein[1] of the sequence $\{s_n\}$ is

$$t_n^{p,q} = \frac{1}{(p*q)_n} \sum_{i=0}^{n} p_{n-i} q_i s_i .$$

The series $\sum_{i=0}^{\infty} a_i$ is said to be summable (N,p,q) to s if $t_n^{p,q} \to s$ as $n \to \infty$.

We shall denote it by $\sum_{i=0}^{n} a_i = s(N,p,q)$ or $s_n \to s(N,p,q)$,

and is said to absolute summable (N,p,q) or summable $|N,p,q|$ if $t_n^{p,q} \in BV$

2. Let $f(t)$ be a periodic function with period 2π and integrable over $(-\pi,\pi)$ without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero so that

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv A_n(t)$$

and $$\int_{-\pi}^{\pi} f(t)dt = 0$$

Let $$\Phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}$$

3. Considering absolute Nörlund summability of a factored Fourier series Singh[2] has proved the following theorem.

Theorem: *If* $\Phi(t)$ *is a function of bounded variation in* $(0,\pi)$ *then the series*

$$\sum \frac{(n+1)p_n}{P_n} A_n(t)$$

is summable $[N, p_n]$, *at* $t = x$, *where the sequence* $\{p_n\}$ *is non-negative and non-increasing such that*

$$\left\{\frac{(n+1)p_n}{P_n}\right\}$$

is of bounded variation.

The aim of this paper is to generalize above theorem for $|N, p, q|$ summability.

We shall prove the following main theorem.

Theorem: *If* $\Phi(t)$ *is a function of bounded variation in* $(0, \pi)$ *then the factored Fourier series.*

$$\sum \frac{(p*q)_n \lambda_n}{n^{\alpha}} A_n(t)$$

is $|N, p, q|$ *summable at* $t = x$ *where the sequences* $\{p_n\}$ *and* $\{q_n\}$ *are non-negative non-increasing such that*

(i) $\left\{\frac{(n+1)p_n}{(p*q)_n}\right\}$ *is of bounded variation*

(ii) $\left\{\frac{(n+1)q_n}{(p*q)_n}\right\}$ *is of bounded variation*

(iii) $\left\{\frac{(p*q)_n}{n^{\alpha}}\right\}$ *is bounded*

(iv) $\left\{\frac{p_{n+1}}{p_n}\right\}$ *is non-decreasing*

(v) $\left\{\frac{q_{n+1}}{q_n}\right\}$ *is non-decreasing*

and $\{\lambda_n\}$ *is a convex sequence such that* $\sum \frac{\lambda_n}{n} < \infty$.

Proof: It is known that if $\Phi_{\alpha}(t), 0 < \alpha \leq$, is of bounded variation the series $\sum A_n(t)$ is convergent. Thus

$$\sum_{n=1}^{m} A_n(t) \frac{(p*q)_n \lambda_n}{n^{\alpha}} = O\left(\sum_{n=1}^{m-1} \left| \Delta \frac{(p*q)_n \lambda_n}{n^{\alpha}} \right| \right) + O\left(\frac{(p*q)_m \lambda_m}{m^{\alpha}}\right)$$

$$= O\left(\sum_{n=1}^{m-1}\frac{\lambda_n}{(n+1)}\left\{\frac{(n+1)p_n q_n}{(p*q)_{n+1}}\right\}\frac{(p*q)_{n+1}}{n^{\alpha}}\right)$$

$$+ O\left(\sum_{n=1}^{m-1}\frac{(p*q)_n\lambda_n}{n^{\alpha+1}}\right) + O\left(\sum_{n=1}^{m-1}\frac{(p*q)_n}{n^{\alpha}}\Delta\lambda_n\right) + O(1)$$

$$= O(1)$$

as $\quad \dfrac{(p*q)_n}{n^{\alpha}} = O(1)$.

To prove the theorem we have simply to show that:

$$\sum\frac{1}{(n+1)(p*q)_{n-1}}\left|\sum_{k=1}^{n} p_{n-k}q_k kA_k(x)\frac{(p*q)_k\lambda_k}{k^{\alpha}}\right| < \infty .$$

Now by Abel's transformation

$$L_n = \left|\sum_{k=1}^{n} p_{n-k}q_k kA_k(x)\frac{(p*q)_k\lambda_k}{k^{\alpha}}\right|$$

$$\le \left|\sum_{k=1}^{n-1}\Delta\left(\frac{(p*q)_k\lambda_k}{k^{\alpha}}\right)\sum_{\mu=1}^{k} p_{n-\mu}q_\mu \mu A_\mu(x)\right| + \frac{(p*q)_n\lambda_n}{n^{\alpha}}\left|\sum_{\mu=1}^{n} p_{n-\mu}\mu A_\mu(x)\right|$$

$$\le \left|\sum_{k=1}^{n-1}\Delta\left(\frac{(p*q)_k\lambda_k}{k^{\alpha}}\right)\right|\left\{\max_{1\le m\le k}\left|\sum_{\mu=1}^{m} p_{m-\mu}\mu A_\mu(x)\right|\right\} + C(p*q)_n\lambda_n$$

$$\le C\sum_{k=1}^{n-1}k^{\alpha}\left|\Delta\left(\frac{(p*q)_k\lambda_k}{k^{\alpha}}\right)\right| + C(p*q)_n\lambda_n$$

$$= L_{n,1} + L_{n,2}$$

Now,

$$\sum_{n=2}^{m}\frac{|L_{n,1}|}{n(p*q)_{n-1}} = \sum_{k=1}^{m-1}k^{\alpha}\left|\Delta\left(\frac{(p*q)_k\lambda_k}{k^{\alpha}}\right)\right|\sum_{n=v+1}^{k}\frac{1}{v(p*q)_{v+1}}$$

$$= \sum_{k=1}^{m-1} \frac{k^{\alpha}}{(p*q)_k} \left| \Delta\left(\frac{(p*q)_k \lambda_k}{k^{\alpha}} \right) \right|$$

$$\leq C \sum_{k=1}^{m-1} \Delta\lambda_k + C \sum_{k=1}^{m-1} \frac{k^{\alpha}(p*q)_k \lambda_k}{(p*q)_k k^{\alpha}} + C \sum_{k=1}^{m-1} \frac{k^{\alpha}}{k^{\alpha+1}} \lambda_k$$

$$\leq C \sum_{k=1}^{m-1} \Delta\lambda_k + C \sum_{k=1}^{m-1} \frac{\lambda_k}{k}$$

$$\leq C,$$

and

$$\sum \frac{1}{n(p*q)_{n-1}} | L_{n,2} | \leq C \sum \frac{\lambda_n}{n} \leq C.$$

This completes the proof of the theorem.

References

1. D. Borwein, On products of sequences, *Jour. London Math. Soc.*, **33** (1958), 352-357.

2. L.B. Singh, Absolute Nörlund summability of a factored Fourier series, *Indian Journal of Mathemtics*, Vol. 12, No. 3 (1970), 189-194.

3. S.N. Bhatt, and Kishore, N., Absolute Nörlund summability of Fourier series, *Indian Jour. Math.* **9** (1967), 259-267.

4. L.Mc. Fadden, Absolute Nörlund summability, *Duke Math. Jour.*, **9** (1942), 168-207.

5. G.H. Hardy, *Divergent series*, Oxford University Press, Oxford (1949).

6. N. Kishore, Absolute Nörlund summability of a factored Fourier series, *Indian Jour. Math.*, **9** (1967), 123-136.

7. T. Singh, Absolute Nörlund summability of a factored Fourier series, *Indian Jour. Math.,* **9** (1967), 227-236.

8. A. Zygmund, *Trigonometric series, Vols. I & II (IInd).* Cambridge University Press, Cambridge (1959)