ISSN 0974 - 9373

Vol. 18 No.3 (2014) **Journal of International Academy of Physical Sciences** pp. 207-212

Special Projective Motions in a Finsler Space

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(Received June 17, 2014)

Abstract: In this paper, we consider a projective motion in a Finsler space whose deviation tensor satisfies certain conditions. It is proved that a projective motion in a Finsler space, whose deviation tensor satisfies $y^k(£B_k H^i_j - B_k £H^i_j) = 0$, is necessarily an affine motion. It is also established that a projective motion which is also a curvature collineation and under which the covariant derivative of the deviation tensor is Lie-symmetric, is either an affine motion or the space is flat. In a recurrent Finsler space, if a projective motion is a curvature collineation and it leaves the recurrence vector invariant, then the projective motion is necessarily an affine motion.

AMS Mathematics Subject Classification: 53B40

Keywords: Projective motion, Deviation tensor, Affine motion, Curvature collineation, Recurrent Finsler space.

1. Introduction

K. Yano and T. Nagano[1] discussed a Riemannian Space $V_n (n > 2)$ admitting a projective motion under which the covariant derivative of Weyl's projective curvature tensor is invariant. They proved that such projective motion is an affine motion if the space is not of constant curvature. They also established that an infinitesimal projective motion in a symmetric Riemannian space $V_n (n > 2)$ is an affine motion if the space is not of constant curvature. P. N. pandey[2] extended these results to an n – dimensional Finsler space $F_n (n > 2)$. He established that an infinitesimal projective motion which leaves the covariant derivative of projective deviation tensor invariant is an affine motion or the space is of scalar curvature. The aim of the present paper is to study an infinitesimal projective motion in a Finsler space, leaving certain tensors invariant.

2. Preliminaries

Let F^n be an n-dimensional Finsler space equipped with a metric function $F(x^i, y^i)$ satisfying the requisite conditions [2, 3]. Let g_{ij}, G^i_{jk} and H^i_{jkh} be the components of the metric tensor, Berwald's connection coefficients and components of Berwald's curvature tensor respectively. The curvature tensor H^i_{jkh} is skew-symmetric in its last two lower indices and positively homogeneous of degree zero in y^i. Transvecting this tensor by y^i we obtain the following tensors:

$$\text{(2.1)} \qquad \text{(a)} \quad H^i_{kh} = H^i_{jkh} y^j , \qquad \text{(b)} \quad H^i_h = H^i_{kh} y^k .$$

The tensor H^i_h satisfies

$$\text{(2.2)} \qquad \text{(a)} \quad \frac{1}{3}\left(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k\right) = H^i_{kh}, \qquad \text{(b)} \quad H^i_i = (\mathrm{n}-1)\,\mathrm{H},$$

$$\text{(c)} \quad y_i H^i_h = 0, \qquad \text{(d)} \quad H^i_h y^h = 0 ,$$

where H is scalar curvature, $y_i = g_{ik} y^k$ and $\dot{\partial}_k = \dfrac{\partial}{\partial y^k}$.

Partial differentiation of connection coefficients G^i_{jk} with respect to y^h yields a tensor being denoted by G^i_{jkh}, i.e $G^i_{jkh} = \dot{\partial}_h G^i_{jk}$. This tensor is symmetric in all its lower indices and satisfies

$$\text{(2.3)} \qquad G^i_{jkh} y^h = 0 .$$

The commutation formula for Barwald covariant differential operator B_k and directional differential operator $\dot{\partial}_h$ is given by

$$\text{(2.4)} \qquad \dot{\partial}_h \mathrm{B}_k T^i_j - \mathrm{B}_k \dot{\partial}_h T^i_j = T^r_j G^i_{hkr} - T^i_r G^r_{hkj} ,$$

where T^i_j is an arbitrary tensor.

A Finsler space F_n is said to be recurrent if there exists a non-zero vector λ_m such that

$$\text{(2.5)} \qquad B_m H^i_{jkh} = \lambda_m H^i_{jkh} , \qquad H^i_{jkh} \neq 0 .$$

The vector λ_m is called the recurrence vector. Pandey [4] proved that the recurrence vector λ_m is independent of directional arguments provided the scalar curvature H does not vanish.

Let us consider an infinitesimal transformation

$$\overline{x}^i = x^i + \in v^i(x^j) \tag{2.6}$$

where ε is an infinitesimal constant and v^i is a contravariant vector field and denote the operator for Lie differentiation with respect to the infinitesimal transformation (2.6) by the symbol £.
The commutation formula for the Lie-differential operator £ and Berwald covariant differential operator B_k is given by

$$£B_k T^i_j - B_k £T^i_j = T^r_j £G^i_{kr} - T^i_r £G^r_{jk} - \left(\dot{\partial}_r T^i_j\right) £G^r_k , \tag{2.7}$$

while the commutation formula for the Lie-differential operator and directional differential operator is given by

$$\dot{\partial}_k £T^i_j - £\dot{\partial}_k T^i_j = 0 \tag{2.8}$$

where T^i_j is an arbitrary tensor.

The infinitesimal transformation (2.6) is called an affine motion if it preserves parallelism of vectors while it is called a projective motion if it preserves geodesics.
The necessary and sufficient condition for the transformation (2.6) to be an affine motion is

$$£G^i_{jk} = 0 , \tag{2.9}$$

while the necessary and sufficient condition for (2.6) to be a projective motion is

$$£G^i_{jk} = y^i p_{jk} + \delta^i_j p_k + \delta^i_k p_j \tag{2.10}$$

where

$$\text{(a)} \quad p_j = \dot{\partial}_j p , \qquad \text{(b)} \quad p_{jk} = \dot{\partial}_j \dot{\partial}_k p , \tag{2.11}$$

p being a scalar invariant positively homogeneous of degree one in y^i. The homogeneity of p implies

$$\text{(a)} \quad p_i y^i = p , \qquad \text{(b)} \quad p_{jk} y^k = 0. \tag{2.12}$$

It is well known that every affine motion is a projective motion. A non-affine projective motion is characterized by (2.10), (2.11) and $p \neq 0$.
The necessary and sufficient condition for the transformation (2.6) to be a curvature collineation is

(2.13) $$\pounds H^i_{jkh} = 0 .$$

3. A Special Projective Motion

Let a Finsler space F^n admits an infinitesimal projective motion (2.6) characterized by (2.10) and (2.11).
Replacing T^i_j in equation (2.7) by H^i_j, we have

(3.1) $$\pounds B_k H^i_j - B_k \pounds H^i_j = H^r_j \pounds G^i_{rk} - H^i_r \pounds G^r_{jk} - \left(\dot{\partial}_r H^i_j\right) \pounds G^r_k ,$$

which, in view of (2.2 d), (2.10) and (2.12), gives

(3.2) $$\begin{aligned} \pounds B_k H^i_j - B_k \pounds H^i_j &= y^i p_{rk} H^r_j + p_r H^r_j \delta^i_k - p_j H^i_k \\ &\quad - 2H^i_j p_k - p \dot{\partial}_k H^i_j \end{aligned} .$$

Transvecting (3.2) by y^k and using (2.2 d), (2.12), we get

(3.3) $$y^k [\pounds B_k H^i_j - B_k \pounds H^i_j] = y^i p_r H^r_j - 4 p H^i_j .$$

Suppose $y^k [\pounds B_k H^i_j - B_k \pounds H^i_j] = 0$, then (3.3) gives

(3.4) $$y^i p_r H^r_j - 4 p H^i_j = 0 .$$

Multiplying (3.4) by y_i $(= g_{ij} y^j)$, and using $y_i y^i = F^2$ and (2.2c), we get

(3.5) $$p_r H^r_j = 0 .$$

Using (3.5) in (3.4), we find $p = 0,$ becasuese the deviation tensor H^i_j of a non-flat Finsler space is non vanishing. This leads to

Theorem 3.1. *A projective motion in a Finsler space* F_n *is an affine motion if the following condition holds*

$$y^k \left[\pounds B_k H^i_j - B_k \pounds H^i_j \right] = 0 .$$

The condition $y^k \left[\pounds B_k H^i_j - B_k \pounds H^i_j \right] = 0$ is obviously satisfied if the Lie differential operator $\pounds$ and Berwald covariant differential operator B_k commute. Thus, we have

Corollary 3.1. *If the operator of Lie- differentiation with respect to a projective motion and Berwald covariant differentiation commute on Berwald deviation tensor, then the projective motion is necessarily an affine motion.*

Suppose the projective motion (2.6) is a curvature collineation, i.e. (2.13) holds. Transvecting (2.13) by $y^j y^k$ and using (2.1), we get

$$\text{(3.6)} \qquad £H^i_h = 0 .$$

In view of (3.6), equations (3.2) becomes

$$\text{(3.7)} \qquad £B_k H^i_j = y^i p_{rk} H^r_j + p_r H^r_j \delta^i_k - p_j H^i_k - 2H^i_j p_k - p\dot{\partial}_k H^i_j .$$

Interchanging the indices j and k in (3.7), we have

$$\text{(3.8)} \qquad £B_j H^i_k = y^i p_{rj} H^r_k + p_r H^r_k \delta^i_j - p_k H^i_j - 2H^i_k p_j - p\dot{\partial}_j H^i_k \quad .$$

Subtracting (3.8) from (3.7) and using (2.2a), we get

$$\text{(3.9)} \qquad \begin{aligned} £\left(B_k H^i_j - B_j H^i_k\right) = \left(H^r_j p_{rk} - H^r_k p_{rj}\right) y^i + \left(H^r_j \delta^i_k - H^r_k \delta^i_j\right) p_r \\ + H^i_k p_j - H^i_j p_k - 3pH^i_{kj} \end{aligned} .$$

Suppose that the tensor $B_k H^i_j - B_j H^i_k = 0$ is Lie-symmetric i.e. $£\left(B_k H^i_j - B_j H^i_k\right) = 0$, then (3.9) gives

$$\text{(3.10)} \qquad \begin{aligned} \left(H^r_j p_{rk} - H^r_k p_{rj}\right) y^i + \left(H^r_j \delta^i_k - H^r_k \delta^i_j\right) p_r + H^i_k p_j \\ - H^i_j p_k - 3pH^i_{kj} = 0. \end{aligned}$$

Transvecting (3.10) by y^k, and using (2.2d) and (2.12), we have

$$\text{(3.11)} \qquad y^i p_r H^r_j - 4pH^i_j = 0 .$$

Transvecting (3.11) by y_i and using (2.2c), we have

$$\text{(3.12)} \qquad p_r H^r_j = 0 ,$$

which together with (3.11) implies $p = 0$ if $H^i_j \neq 0$ and $H^i_j = 0$ if $p \neq 0$. This leads to

Theorem 3.2. *A projective motion in a Finsler space* F_n *which is a curvature collineation and with respect to which the covariant derivative of deviation tensor is Lie-symmetric, is either an affine motion or the space* F_n *is flat.*

If the Finsler space is a recurrent space characterized by (2.5), the condition

(3.13) $$£\left(B_k H^i_j - B_j H^i_k\right) = 0,$$

may be written as

(3.14) $$\left(£\lambda_k\right) H^i_j - \left(£\lambda_j\right) H^i_k = 0.$$

Transvecting (3.14) by y^k, we have

$$£\left(\lambda_k y^k\right) H^i_j = 0,$$

which implies $£\left(\lambda_k y^k\right) = 0$ for $H^i_j \neq 0$. Differentiating $£\left(\lambda_k y^k\right) = 0$ partially with respect to y^j, we get $£\lambda_j = 0$. Conversely, $£\lambda_j = 0$ implies (3.14) and hence (3.13). Thus, (3.13) is equivalent to $£\lambda_j = 0$. In view of this and theorem 3.2, we conclude

Theorem 3.3. *A projective motion in a recurrent Finsler space, which is a curvature collineation and leaves the recurrence vector invariant, is necessarily an affine motion.*

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